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A covering theorem for the core model below a Woodin cardinal

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DEDICATION

To my family for their years of support and guidance.

To Nikki, Harley, and Luna for bringing me happiness.

TABLE OF CONTENTS

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ABSTRACT OF THE DISSERTATION

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The main result of this dissertation is a covering theorem for the core model below a Woodin cardinal. More precisely, we work with Steel's core model K constructed in V_Ω where Ω is measurable. The theorem is in a similar spirit to theorems of Mitchell and Cox and roughly says that either K recognizes the singularity of an ordinal κ or else κ is measurable in K.

The first chapter of the thesis builds up the technical theory we will work in. The premice we work with use Mitchell-Steel indexing, but we use Jensen's Σ^* fine structure and a different amenable coding. The use of Σ^* fine structure and this amenable coding significantly simplifies the theory. Towards the end of the first chapter, we prove the full condensation lemma for premice with Mitchell-Steel indexing. This was originally proven by Jensen for premice with λ-indexing. The second chapter is devoted to the proof of the above mentioned covering theorem.

Introduction

This work is a contribution to the area of set theory known as inner model theory. One of the goals of inner model theory is to construct canonical inner models of ZFC which can contain large cardinals. These models are often of the form $\mathsf{L}[E]$ where E is a coherent sequence of extenders, and are known as *extender models*. The initial segments of extender models are known as *premice*. The extenders on the sequence E code elementary embeddings (defined on $\mathsf{L}[E]$ or its premice) and is what ensures the model will have large cardinals. However, these sequences are constructed in a precise manner so that these models have an L-like fine structure and can be studied level by level. This is one of the ways that these models are "canonical"—the fine structure allows us to analyze when new sets are added to the model. Moreover, if these models are *iterable* there is a nautral way to compare them. That is, given two models constructed relative to extender sequences E and E' there is a way to determine which model has more information.

In the 1970's, Jensen [3] proved the covering lemma for L, which says that either $0^\#$ exists or else every uncountable set of ordinals x can be covered by a set $y \in L$ of the same cardinality. In the absence of $0^\#$, L is a good approximation of V and this result led to the notion of a *core model*.

The core model K is an extender model which approximates V relatively well. However, the existence of K depends on the large cardinal structure of the universe. If K does exist, it has the following properties:

- It is *generically absolute*: $\mathsf{K}^{\mathrm{V}[G]} = \mathsf{K}^{\mathrm{V}}$ whenever G is V-generic for a set sized forcing
- It is *rigid*: there is no nontrivial embedding $j : \mathsf{K} \to \mathsf{K}$
- It satisfies *weak covering*: for any K-cardinal $\kappa \geq \omega_2$, if $\lambda = \kappa^{+K}$, then $\mathrm{cf}(\lambda) \geq \mathrm{card}(\kappa)$

K is a relative notion and depends on the universe it is constructed in. For example, if $0^{\#}$ does not exist then $\mathsf{K} = \mathsf{L}$. If K exists in a universe, then it will be maximal in that universe in the sense that it absorbs the large cardinals found there. The construction of a core model is always under a suitable anti-large cardinal hypothesis. Indeed, it is the anti-large cardinal hypothesis which enables the proof of the above properties and ensures the core model is a good approximation to V.

Core models have been constructed under increasingly weaker anti-large cardinal hypothesis. Dodd and Jensen [4], constructed the core model under the assumption there is no inner model with a measurable cardinal. Mitchell [10], constructed the core model under the assumption there is no inner model with a measurable cardinal κ with $o(\kappa) = \kappa^{++}$. Later, Jensen [5] constructed the core model assuming the sharp for a strong cardinal does not exist. Steel constructed core model below a Woodin cardinal in [14], under the technical assumption that there is a measurable cardinal Ω. Later work of Jensen and Steel (cf. [6]) showed how to remove this technical assumption.

As mentioned above, Jensen discovered the original covering lemma for L and with it a sort of dichotomy. If $0^{\#}$ does not exist, then L is a good approximation of V. However, if $0^{\#}$ does exist, then L is much thinner than V. Indeed, each uncountable V-cardinal will be inaccessible inside L (and more), and each L-successor has countable cofinality in V.

Dodd and Jensen proved that their core model satisfies the same strong form of covering assuming there is no inner model with a measurable cardinal. Past a measurable cardinal, this strong form of covering for the core model cannot be proved as Prikry forcing shows. However, by adding (if necessary) a Prikry sequence C, Dodd and Jensen were able to prove the strong covering property for the model $L[U]$ assuming the sharp for a measurable cardinal does not exist. Mitchell [8] obtained a similar (but more complicated) result for the core model for sequences of measures, where the Prikry sequence C was replaced by a *system of indiscernibles* $\mathcal{C}$. Additionally in [8], Mitchell proved another result with a dichotomous nature:

Theorem (Mitchell, 1987)**.** *Assume there is no inner model satisfying* $(\exists\kappa)\ o(\kappa) = \kappa^{++}$*, and that* κ *is a singular cardinal which is regular in* K*. Suppose* $\omega < \delta = \mathrm{cf}(\kappa)$ *and* $\delta^\omega < \kappa$*. Then,* $o^{\mathsf{K}}(\kappa) \geq \delta$*.*

Here and below, $o^{\mathsf{K}}(\kappa)$ denotes the Mitchell order of κ inside K. This theorem roughly says that either K recognizes the singularity of κ, or else κ is measurable in K. Later, working with a larger core model, Cox (cf. [2]) proved:

Theorem (Cox, 2009)**.** *Let* K *be the core model below the sharp for a strong cardinal. Assume* $\omega_2 < \kappa$*,* κ *is regular in* K *and* $\mathrm{cf}(\kappa) < \mathrm{card}(\kappa)$*. Then,* κ *is measurable in* K*. Moreover, if* $\mathrm{cf}(\kappa) > \omega$*, then* $o^{\mathsf{K}}(\kappa) \geq \mathrm{cf}(\kappa)$*.*

The main result of this dissertation is in a similar vein, working with Steel's core model below a Woodin cardinal, but with additional cardinal arithmetic assumptions:

Theorem (Main Theorem)**.** *Let* K *be the core model below a Woodin cardinal constructed in* V_Ω *where* Ω *is measurable. Assume* $\kappa > \omega_2$ *is a regular cardinal in* K*, but* $\mathrm{cf}(\kappa) < \mathrm{card}(\kappa)$*. Let* $\gamma = \mathrm{cf}(\kappa)$ *and assume* $\mathrm{card}(\kappa)$ *is* γ*-closed. Then,* κ *is measurable in* K*. Moreover, if* $\gamma > \omega$*, then* $o^{\mathsf{K}}(\kappa) \geq \gamma$*.*

This theorem is the conjunction of Theorems 2.3 and 2.22 which will be proved in Chapter 2. The author has very recently found out that Mithell and Schimmerling have announced that they can prove this theorem without the cardinal arithmetic assumption.

The first chapter of the dissertation reviews the technical background that is necessary for the later exposition. The premice we work with index extenders according to Mitchell-Steel indexing as in [13], but we use a different amenable coding and Jensen's Σ^* fine structure. We code our premice into amenable structures using the Jensen expansion. The use of this amenable coding and Σ^* fine structure greatly simplifies the theory and allows us to treat premice of different types in a more uniform way. In Section 1.5, we prove the full condensation lemma for premice with Mitchell-Steel indexing. This was originally proved by Jensen for premice with λ-indexing. Chapter 2 proves the main results of the dissertation, namely Theorems 2.3 and 2.22.

Chapter 1

Preliminaries

1.1 Review of Fine Structure

In this section we review some of the basic fine structural definitions and propositions that will be relevant in later sections. We follow Jensen's Σ^* fine structure as detailed in [16], [15], and [7].

Convention: Whenever we say that $M = (J_\alpha^A, B)$ is a J-*structure* we are assuming that the predicates A, B are amenable.

Definition 1.1. A J-structure $M = (J_\alpha^A, B)$ is *acceptable* iff whenever $\xi < \alpha$ and $\mathcal{P}(\tau) \cap J_{\xi+1}^A \not\subseteq J_\xi^A$ for some $\tau < \omega\xi$, there is a surjective map $f : \tau \to \omega\xi$ in $J_{\xi+1}^A$.

Definition 1.2. For an acceptable J-structure $M = (J_\alpha^A, B)$, we inductively define

i. the n^{th}-*projectum* of M is

$$\rho_M^0 = \alpha$$
$$\rho_M^{n+1} = \min\{\rho \leq \alpha \mid \text{for some } \mathbf{\Sigma}_1^{(n)}(M)\text{-relation } A, A \cap \rho \notin M\}$$
$$\rho_M^\omega = \min\{\rho_M^n \mid n < \omega\}$$

ii. the *uniformly good* $\Sigma_1^{(n)}(M)$ Skolem functions $\tilde{h}_M^n$. Given $p \in [\mathrm{On} \cap M]^{<\omega}$

$$\tilde{h}_M^0(z, p) \simeq h_{M^{n,p}}((z)_0, \langle (z)_1, p\rangle)$$
$$\tilde{h}_M^{n+1}(z, p) \simeq \tilde{h}_M^n((z)_0, h_{M^{n,p}}((z)_1, \langle (z)_2, p^{(n+1)}\rangle))$$

iii. the set P_M^n of *good parameters*

$$P_M^0 = [\mathrm{On} \cap M]^{<\omega}$$
$$P_M^{n+1} = \{p \in P_M^n \mid \text{for some } \Sigma_1^{(n)}(M)\text{-relation } A \text{ in } p, A \cap \rho_M^{n+1} \notin M\}$$
$$P_M^* = \{p \in M \mid (\forall n < \omega)\ p \in P_M^n\}$$

iv. the set R_M^n of *very good parameters*

$$R_M^0 = P_M^0$$
$$R_M^{n+1} = \{p \in R_M^n \mid \tilde{h}_M^{n+1}(\rho_M^{n+1} \cup \{p\}) = M\}$$
$$R_M^* = \{p \in M \mid (\forall n < \omega)\ p \in R_M^n\}$$

Definition 1.3. An acceptable J-structure $M = (J_\alpha^A, B)$ is n-*sound* iff $R_M^n = P_M^n$, and it is *sound* iff it is n-sound for all $n < \omega$.

Definition 1.4. Let $M = (J_\alpha^A, B)$ be acceptable and $<^*$ the lexicographical ordering on finite sets of decreasing ordinals. The *standard parameter above* ρ_M^n is the $<^*$-least $p \in P_M^n$

and is denoted by p^n_M. The *standard parameter* of M is the $<^*$-least $p \in P^*_M$ and is denoted by p_M.

Lemma 1.5. *Let $M = (J^A_\alpha, B)$ be acceptable. Then, M is n-sound iff $p^k_M \in R^k_M$ for all $k \leq n$. Similarly, M is sound iff $p^n_M \in R^n_M$ for all $n < \omega$.*

Remark 1.6. If good parameters can be lengthened (which will be true in most cases we are concerned about), then the previous lemma can be simplified to only having to check $p^n_M \in R^n_M$ for n-soundness and $p_M \in R^*_M$ for soundness.

We list here the definition of *solidity* but leave out some technical details. which can be found on pages 41 - 43 of [16].

Definition 1.7. Let $M = (J^A_\alpha, B)$ be acceptable. Then, M is *solid above* α iff the standard witness $W^{\nu,p_M} \in M$ for all $\nu \in p_M - \alpha$. We say M is *solid* iff $W^{\nu,p_M} \in M$ for all $\nu \in p_M$. If M is solid above ρ^n_M, then we say M is n-solid.

Solidity is important for a few reasons, one of which is that it guarantees that standard parameters are mapped to each other under sufficiently preserving maps.

Lemma 1.8. *Let $\bar{M}, M$ be acceptable and $\sigma : \bar{M} \to M$ be $\Sigma^{(n)}_1$-preserving. Suppose $\bar{p} \in P^{n+1}_{\bar{M}}$, $p = \sigma(\bar{p}) \in P^{n+1}_M$, $\bar{\alpha} \geq \rho^{n+1}_{\bar{M}}$, $\alpha = \sigma(\bar{\alpha}) \geq \rho^{n+1}_M$ and $\bar{M}$ is solid above α. Then, $\bar{p} - \bar{\alpha} = p^{n+1}_{\bar{M}} - \alpha = p_{\bar{M}} - \bar{\alpha}$, $p - \alpha = p^{n+1}_M - \alpha = p_M - \alpha$ and M is solid above α.*

The following lemmas can be thought of a Downward Extensions of Embeddings lemma in the Σ^*-context.

Lemma 1.9. *Let $M = (J^\alpha_A, B)$ be acceptable and assume $X \subseteq M$ is closed under good $\mathbf{\Sigma}^{(n)}_1(M)$ functions and $X \cap P^n_M \neq \emptyset$. Let $\bar{M}$ be the transitive collapse of X and $\sigma : \bar{M} \to M$ the inverse of the collapsing map. Then, σ is $\Sigma^{(n)}_1$-preserving.*

Remark 1.10. Often, we do not have such an X directly, but instead we will form one by looking at the hull $\tilde{h}^{n+1}_M(Z \cup \{p^{n+1}_M\})$ of some set $Z \subseteq M$.

Lemma 1.11. *Let $M = (J^{\alpha}_{A}, B)$ and $\bar{M} = (J^{\bar{\alpha}}_{\bar{A}}, \bar{B})$ be acceptable. Assume $\sigma : \bar{M} \to M$ is $\Sigma^{(n)}_1$-preserving where n is such that $\sigma \restriction \rho^{n+1}_M = \mathrm{id}$ and $\mathrm{rng}(\sigma) \cap P^*_M \neq \emptyset$. Then, σ is Σ^*-preserving.*

1.2 Extenders and Ultrapowers

An extender is simply a way to code an elementary embedding into a set sized object. Here we quickly review the defintion of an extender and list some preservation properties of fine ultrapowers.

Definition 1.12 (Hypermeasure representation of an extender)**.** Suppose M is transitive and rudimentarily closed and $\kappa < \lambda$. We call E a (κ, λ)-*extender over* M iff there is a nontrivial Σ_0-elementary embedding $j : M \to N$ with with N transitive and rudimentarily closed, such that $\kappa = \mathrm{cr}(j)$ and $\lambda \leq j(\kappa)$ and

$$E = \{(a, x) \mid a \in [\lambda]^{<\omega}, x \in \mathcal{P}([\kappa]^{|a|}) \cap M \text{ and } a \in j(x)\}$$

As is common, given $a \in [\lambda]^{<\omega}$ we will use E_a to denote the ultrafilter on $\mathcal{P}([\kappa]^{|a|}) \cap M$ defined by $E_a = \{x \in [\kappa]^{|a|} \mid (a, x) \in E\}$.

Definition 1.13. We call κ the *critical point* of the extender and write $\mathrm{cr}(E) = \kappa$ and we call λ the *length* of the extender and write $\mathrm{lh}(E) = \lambda$.

Definition 1.14. Let E be a (κ, λ)-extender over M and $\xi < \lambda$. We will write $E \restriction \xi$ for the extender defined by

$$E \restriction \xi = \{(a, x) \mid a \in [\xi]^{<\omega} \text{ and } (a, x) \in E\}$$

We call ξ a *generator* of E iff for every $a \in [\xi]^{<\omega}$ and every $f : [\kappa]^{|a|} \to M$ with $f \in M$

$$\{u^\frown\eta \in [\kappa]^{|a|+1} \mid f(u) \neq \eta\} \in E_{a\cup\{\xi\}}$$

This says that $\xi \neq [a, f]_E$ for any a, f as above.

Definition 1.15. Given an extender E over M, we set

$$\nu(E) = \sup\left(\{\xi + 1 \mid \xi \text{ is a generator}\} \cup \kappa^{+M}\right)$$

and we call $\nu(E)$ the *natural length* of E. Let $M' = \mathrm{Ult}^0(M, E)$ and $\pi : M \to_E M'$ and $\alpha = \nu(E)^{+M'}$. The *trivial completion* of E is the extender G of length α derived from π, that is for $a \in [\alpha]^{<\omega}$ and $x \in \mathcal{P}([\mathrm{cr}(E)]^{|a|}) \cap M$

$$(a, x) \in G \longleftrightarrow a \in \pi(x)$$

Next we review some preservation properties associated with fine ultrapowers. In the following, let $M = (J^A_\alpha, B)$ be acceptable and F an extender over M. The proofs of all of the following can be found in [16].

Lemma 1.16. *Assume F is a weakly amenable extender and $N = \mathrm{Ult}^*(M, F)$. Let n be such that $\rho^{n+1}_M \leq \mathrm{cr}(F) < \rho^n_M$. Then, π is $\Sigma^{(n)}_0$-cofinal, so* $\sup \pi[\rho^n_M] = \rho^n_N$.

Lemma 1.17. *Assume F is a extender and $N = \mathrm{Ult}^*(M, F)$. Let n be such that $\rho^{n+1}_M \leq \mathrm{cr}(F) < \rho^n_M$. Assume that $R^n_M \neq \emptyset$. Then, π is $\Sigma^{(n)}_0$-cofinal and $\pi[R^n_M] \subseteq R^n_N$.*

Lemma 1.18. *Assume F is an extender and $N = \mathrm{Ult}^k(M, F)$. Assume that $R^k_M \neq \emptyset$. Then, π is $\Sigma^{(k)}_0$-cofinal and $\pi[R^k_M] \subseteq R^k_N$.*

Lemma 1.19. *Assume that F is close to M and $N = \mathrm{Ult}^*(M, F)$. Then, for each $m < \omega$ such tht $\rho^m_M \leq \mathrm{cr}(F)$*

(a) $H^m_M = H^m_N$

(b) $\mathbf{\Sigma}^{(m)}_1(M) \cap \mathcal{P}(H^m_M) = \mathbf{\Sigma}^{(m)}_1(N) \cap \mathcal{P}(H^m_N)$

Moreover, $\pi : M \to N$ is fully Σ^*-preserving.

1.3 Coherent Structures and Premice

The premice we use are similar to the premice defined in [13]. In particular, we use Mitchell-Steel indexing of extenders. However, we require a weaker initial segment condition and use Σ^* fine structure. Moreover, instead of using the amenable coding used in [13] we code our premice using *coherent* structures. The use of Σ^* fine structure and our amenable coding allows us to treat premice of different types in a more uniform way.

Before getting to the definition of coherent structure, we review the map representation of an extender. While the extenders on the sequence of our premice, will use the hypermeasure representation, the top extender of our coded premice will be viewed as a map.

Definition 1.20 (Map representation of an extender)**.** Suppose M is a transitive ZFC^- model and κ is the largest cardinal of M. Then, F is a *whole* (κ, λ)*-extender over* M iff there is an elementary embedding $\pi : M \to M'$ with $\mathrm{cr}(\pi) = \kappa$ and $\lambda = \pi(\kappa)$ and such that

- F is a function with $\mathrm{dom}(F) = \mathcal{P}(\kappa) \cap M$
- Every element of M' is of the form $\pi(f)(\alpha)$ for some $\alpha < \lambda$ and $f \in M$ with domain κ
- $F(x) = \pi(x) \cap \lambda$ for $x \in \mathcal{P}(\kappa) \cap M$

Remark 1.21*.* An extender is *whole* iff its length is equal to $\pi(\kappa)$. As we will only deal with whole extenders when looking at the map representation we included $\lambda = \pi(\kappa)$ in the definition. However, replacing $\lambda = \pi(\kappa)$ with $\lambda \leq \pi(\kappa)$ will give a perfectly valid definition.

Definition 1.22. A structure $M = (J^E_\alpha, F)$ is *coherent* iff J^E_α is acceptable and F is a whole extender on $J^E_{\bar\alpha}$ for some $\bar\alpha < \alpha$ such that

- $\kappa = \mathrm{cr}(F)$ is the largest cardinal in $J^E_{\bar\alpha}$
- $J^E_\alpha = \mathrm{Ult}^0(J^E_{\bar\alpha}, F)$

Remark 1.23. When working with a coherent structure $M = (J^E_\alpha, F)$ we will always consider F in its map representation.

Coherent structures are ZFC^- models and have a few important properties as outlined in the next lemma

Lemma 1.24. *1. There is a Q-formula ψ such that $M = (J^E_\alpha, F)$ is a coherent structure iff $M \models \psi$*

2. Every coherent structure is amenable

3. M has a largest cardinal $\lambda = F(\mathrm{cr}(F))$

4. If $M = (J^E_\alpha, F)$ is a coherent structure then F is weakly amenable iff $\bar\alpha = \mathrm{cr}(F)^+$, where $\bar\alpha$ is such that $J^E_\alpha = \mathrm{Ult}(J^E_{\bar\alpha}, F)$. Equivalently, F is weakly amenable iff $\mathrm{dom}(F) = \mathcal{P}(\kappa) \cap J^E_\alpha = \mathcal{P}(\kappa) \cap J^E_{\kappa^{+M}}$ where $\kappa = \mathrm{cr}(F)$.

Definition 1.25. A *potential premouse* is an acceptable structure $M = (J^E_\alpha, E_\alpha)$ such that for all $\beta \in \mathrm{dom}(E)$, $E_\beta = \emptyset$ or E_β is a weakly amenable (κ, β)-extender over J^E_β for some κ such that $J^E_\beta \models \kappa^+$ exists and

1. E_β is the trivial completion of $E_\beta \restriction \nu(E_\beta)$ and E_β is not of type Z
2. (Coherence) Letting $\pi : J^E_\beta \to_{E_\beta} J^{E'}_{\beta'}$ be the coarse ultrapower map, we have $E' \restriction \beta = E \restriction \beta$ and $E'_\beta = \emptyset$
3. (Weak ISC) If η is such that $\kappa^{+J^E_\beta} \leq \eta < \nu(E_\beta)$, then $E_\beta \restriction \eta \in J^E_\beta$

Definition 1.26. For a potential premouse $M = (J_\alpha^E, E_\alpha)$ we say M is *active* iff $E_\alpha \neq \emptyset$. Otherwise, M is *passive*.

Definition 1.27. Given a potential premouse $M = (J_\alpha^E, E_\alpha)$ and $\zeta < \alpha$ we set $M||\zeta = (J_\zeta^E, E_\zeta)$ and $M|\zeta = J_\zeta^E$.

Definition 1.28. Given a potential premouse $M = (J_\alpha^E, E_\alpha)$, we code the potential premouse into an amenable structure known as the *expansion* of M and denoted by $\widehat{M}$. If M is passive, then $\widehat{M} = M$. If M is active, then letting $\kappa = \text{cr}(E_\alpha), \tau = \kappa^{+M}$ and $\pi : J_\tau^E \rightarrow_{E_\alpha} J_\beta^{\hat{E}}$ then

$$\widehat{M} = (J_\beta^{\hat{E}}, \hat{E}_\alpha) \qquad \text{where } \hat{E}_\alpha = \pi \restriction (\mathcal{P}(\kappa) \cap M)$$

Remark 1.29. Notice, that in the case that M is an active potential premouse, then its expansion $\widehat{M}$ is a coherent structure and satisfies all the properties of Lemma 1.24

Note that to do any fine structure, we need the model we are considering to be amenable. So, any time we talk about the fine structure of a potential premouse M, we are actually referring to the fine structure of its expansion $\widehat{M}$. Hence, we adopt the following convention

Convention: Suppose M is a potential premouse. Whenever we refer to any fine structural objects of M (e.g. $\rho_M^n, p_M^n, \tilde{h}_M^n$, etc.) we are actually referring to those objects as defined over $\widehat{M}$. If we say that M is n-sound, then we actually mean that $\widehat{M}$ is n-sound. Moreover, when we take ultrapowers of M, we are actually taking ultrapowers of $\widehat{M}$ and similarly, if we have another potential premouse M' then by "an embedding from M to M'" we would literally mean an embedding between their expansions, i.e. an embedding from $\widehat{M}$ to $\widehat{M'}$.

Definition 1.30. A *premouse* is a potential premouse $M = (J_\alpha^E, E_\alpha)$ such that every proper initial segment of M is sound. That is, for every $\zeta < \alpha, M||\zeta$ is sound (note that following the above convention we literally mean that $\widehat{M||\zeta}$ is sound).

Remark 1.31. It is a fact (cf. [16]) that there is a recursive sequence of sentences $(\psi_n \mid n < \omega)$ such that an acceptable structure P is sound iff $(\forall n)\, P \models \psi_n$.

Definition 1.32. Let $M = (J_\alpha^E, E_\alpha)$ be a premouse. We say M is

- *type I* iff $\nu(E_\alpha) = \kappa^{+M}$
- *type II* iff $\nu(E_\alpha)$ is a successor ordinal
- *type III* iff $\nu(E_\alpha)$ is a limit ordinal $> \kappa^{+M}$

Remark 1.33. Following [16], if a premouse M is type II, then we have a fixed constant symbol e which represents the cutback of E_{top}^M by its largest generator. We will always suppress this parameter, but it should be understood that the constant symbol is used when talking about the fine structure of type II premouse or embeddings between type II premouse.

For this section, we will work explicity with the expansion of premice to prove some basic properties. For example, we are going to prove that given the expansion N of a premouse and a weakly amenable extender G over N, the ultrapower $\mathrm{Ult}^k(N, G)$ is the expansion of a premouse of the same type as N. In later sections, we often do not explicitly distinguish between a premouse and its expansion.

Remark 1.34. Note, that if $M = (J_\alpha^E, E_\alpha)$ is an active premouse and N is the expansion of M, then by coherence, $M|\alpha = N|\alpha$. However, $M \neq N||\alpha$, as $E_\alpha^N = \emptyset$.

Remark 1.35. If $N = (J_\beta^E, F)$ is the expansion of some premouse, then the premouse coded by N can be obtained as follows. Let $\alpha = \nu(F)^{+N}$. Then, N is the expansion of $M = (J_\alpha^E, E_\alpha)$ where E_α is the (κ, α)-extender derived from F.

First, we give some lemmas that will be useful for the computations to come.

Lemma 1.36. *Let $N = (J_\beta^E, F)$ be a coherent structure and $\kappa = \mathrm{cr}(F), \lambda = \mathrm{lh}(F)$. Then,*

(a) For $x \in \mathcal{P}(\kappa) \cap N$, the statement "$z = F(x)$" is uniformly $\Delta_1(N)$.

(b) The ordinals κ and λ are uniformly $\Delta_1(N)$.

Proof. For (a), fix $x \in \mathcal{P}(\kappa) \cap N$ and notice

$$z = F(x) \longleftrightarrow (\exists y)\ ((x, y) \in F \wedge z = y) \longleftrightarrow (\forall y)\ ((x, y) \in F \rightarrow z = y)$$

For (b), note that (κ, λ) are the unique $(x, y) \in F$ such that x and y are both ordinals, so

$$x = \kappa \longleftrightarrow (\exists y)\ (x, y \in \mathrm{On} \wedge (x, y) \in F) \longleftrightarrow (\forall y)\ (x \in \mathrm{On} \wedge (x, y) \in F \rightarrow y \in \mathrm{On})$$

In a similar manner we can show that the statement "$z = \lambda$" is $\Delta_1(N)$. $\square$

Lemma 1.37. *Suppose M and M' are two premice of the same type and let $N = (J^E_\beta, F)$ and $N' = (J^{E'}_{\beta'}, F')$ be their expansions, respectively. Let $\kappa = \mathrm{cr}(F), \tau = \kappa^{+N}$, $\kappa' = \mathrm{cr}(F')$, and $\tau' = \kappa'^{+N'}$. Suppose $\sigma : J^E_\tau \rightarrow_F J^E_\beta$ and $\sigma' : J^{E'}_{\tau'} \rightarrow_{F'} J^{E'}_{\beta'}$ denote the expansion ultrapower maps. If $\pi : N \rightarrow N'$ is Σ_1 preserving, then*

$$\pi \circ \sigma(x) = \sigma' \circ \pi(x) \qquad \textit{for all } x \in J^E_\tau \tag{1.1}$$

Proof. The main point here is that the statment "$\sigma(x) = y$" is uniformly $\Sigma_1(N)$ for coherent structures N. To see this, notice

$$\sigma(x) = y \longleftrightarrow (\exists \xi, \eta)(\exists f : \kappa \overset{\text{onto}}{\rightarrow} J^E_\xi)(\exists Z)[x \in J^E_\xi \wedge f(\eta) = x \wedge Z = F(X_f) \wedge g_Z(\eta) = y]$$

Here, we are encoding the function f into the set $X_f \subseteq \kappa$ as

$$X_f = \{\langle\!\langle \zeta_1, \zeta_2 \rangle\!\rangle \mid f(\zeta_1) \in f(\zeta_2)\}$$

and we decode the set $Z = F(X_f) \subseteq \lambda = \mathrm{lh}(F)$ as the function

$$g_Z(\zeta_2) = \{g_Z(\zeta_1) \mid \langle\!\langle \zeta_1, \zeta_2 \rangle\!\rangle \in Z\}$$

where $\langle , \rangle$ denotes the Gödel pairing function.

Let $\varphi(u, v)$ denote the above Σ_1 formula. For $x \in J^E_\tau$, because π is Σ_1-elementary, we have

$$\begin{aligned} N \models \sigma(x) = y &\iff N \models \varphi(x, y) \\ &\iff N' \models \varphi(\pi(x), \pi(y)) \\ &\iff N' \models \sigma'(\pi(x)) = \pi(y) \end{aligned}$$

□

Now, we check that if we apply an extender to the expansion of a premouse (following the rules used in an iteration), then the ultrapower is again an expansion of a premouse of the same type.

Lemma 1.38. *Let M be a type I premouse and $N = (J^E_\beta, F)$ the expansion of M. Assume G is a weakly amenable extender with respect to M and $\mu = \mathrm{cr}(G) < \rho^k_M$ (in particular $\mu < \alpha$). Then, $N' = \mathrm{Ult}^k(N, G)$ is the expansion of a type I premouse.*

Proof. Let $N' = (J^{E'}_{\beta'}, F')$ and set $\kappa = \mathrm{cr}(F), \tau = \kappa^{+N}, \lambda = \mathrm{lh}(F), \nu = \nu(F)$. Let $\pi : N \to N'$ denote the ultrapower map and set $\kappa' = \pi(\kappa)$, $\tau' = \pi(\tau)$, $\lambda' = \pi(\lambda)$, and $\nu' = \pi(\nu)$. We will show that $\kappa' = \mathrm{cr}(F'), \lambda' = \mathrm{lh}(F'), \tau' = \kappa'^{+N'}$ and $\nu' = \nu(F')$. We consider two cases based on what k.

Case 1: Assume $k > 0$.

Then, π is Σ_2-preserving, so N' is also a coherent structure as being coherent is a Q-condition. It follows that $\kappa' = \mathrm{cr}(F'), \lambda' = \mathrm{lh}(F')$ and $\tau' = \kappa'^{+N'}$. Note for a coherent structure being a generator of the top extender is Π_1. For example, N says ξ is a generator iff

$$N \models (\forall f : \kappa \to \kappa)(\forall \beta < \xi)(\forall y)[y = F(\{\langle \eta, \eta' \rangle < \kappa \mid f(\eta) \neq \eta'\}) \longrightarrow \langle \beta, \xi \rangle \in y]$$

To see this is $\Pi_1(N)$ recall that by Lemma 1.36 the statement "$y = F(x)$" is $\Delta_1(N)$. It follows that for a coherent structure the statement "ξ is the only generator of F" is Π_2. Because N is type I,

$$N \models \kappa \text{ is the only generator of } F$$

As $\kappa' = \pi(\kappa)$ and π is Σ_2-preserving, we get

$$N' \models \kappa' \text{ is the only generator of } F' \tag{1.2}$$

Moreover, F' is weakly amenable. To see this, note that it suffices to show that $\operatorname{dom}(F') = \mathcal{P}(\kappa') \cap N'$. Since F is weakly amenable,

$$N \models (\forall x)(\exists y)\ x \subseteq \kappa \to (x, y) \in F$$

As this is a Π_2 statement it will be preserved by π, so

$$N' \models (\forall x)(\exists y)\ x \subseteq \kappa' \to (x, y) \in F'$$

Thus, $\operatorname{dom}(F') = \mathcal{P}(\kappa') \cap N'$. We have now shown that $\nu' = \nu(F')$ and that F' is weakly amenable with respect to N'.

Let $M' = (J^{E'}_{\alpha'}, E_{\alpha'})$ where $\alpha' = \nu'^{+N'}$ and $E_{\alpha'}$ is the trivial completion of $F' \restriction \nu'$. Then, $E_{\alpha'}$ is weakly amenable with respect to M' because F' is and it satisfies the coherency condition of being a potential premouse as N' is a coherent structure. Furthermore, M' is type I by Equation 1.2 so it trivially satisfies the initial segment condition. Thus, M' is a potential type I premouse.

For premousehood, we just need to check that its proper initial segments are sound. Recall that there is a recursive sequence of sentences $(\psi_n \mid n < \omega)$ such that if P is an acceptable structure then P is sound iff $(\forall n < \omega)\ P \models \psi_n$. Because M is a premouse, for $\zeta < \alpha$ the proper initial segments $M||\zeta = N||\zeta$ are sound, (truly $\widehat{N||\zeta}$ is sound). Additionally, $\widehat{N||\zeta} \in J^E_\lambda$ as λ is inaccessible in N, and the satisfaction relation for $\widehat{N||\zeta}$ is definable over J^E_λ. So,

$$N \models (\forall \zeta < \alpha)(\forall n < \omega)\ \widehat{N||\zeta} \models \psi_n$$

By elementarity,

$$N' \models (\forall \zeta < \pi(\alpha))(\forall n < \omega)\ \widehat{N'||\zeta} \models \psi_n$$

Note, that $\pi(\alpha) = \alpha'$ as $\alpha = \tau^{++N}$ and $\alpha' = \tau'^{++N'}$. As $N'|\alpha' = M'|\alpha'$, this proves that all proper initial segments of M' are sound. Hence, N' is the expansion of a type I premouse. $\square$ Case 1

Case 2: Assume $k = 0$.

In this case we know π is Σ_0-cofinal. It follows again that N' is a coherent structure as Σ_0-cofinal maps preserve Q-conditions. Hence, we again obtain $\kappa' = \mathrm{cr}(F'), \lambda' = \mathrm{lh}(F')$ and $\tau' = \kappa'^{+N'}$. We do not have enough preservation to argue that κ' is the only generator of F' in the way we did before, so we take a different approach.

Let $\sigma : J^E_\tau \to_F J^E_\beta$ and $\sigma' : J^E_{\tau'} \to_{F'} J^{E'}_{\beta'}$ be the expansion ultrapower maps. We will show that any $\xi < \lambda'$ can be written as

$$\xi = \sigma'(f)(a) \qquad \text{for some } a \in [\tau']^{<\omega} \text{ and } f : [\kappa']^{|a|} \to N' \text{ with } f \in N' \tag{1.3}$$

Hence, if $\xi > \tau'$ then ξ is not a generator of F', which shows $\nu' = \nu(F')$.

As $\xi \in N'$, there is some $h : \mu \to N$, $h \in N$ and $b \in [\mathrm{lh}(G)]^{<\omega}$ such that $\xi = \pi(h)(b)$ As N

is a coherent structure and F is type I, it follows that $h = \sigma(g)(\kappa)$ for some $g : \kappa \to J^E_\tau$. So, we have

$$\begin{aligned}\xi = \pi(h)(b) = \pi(\sigma(g)(\kappa))(b) &= \pi \circ \sigma(g)(\pi(\kappa))(b) \\ &= \sigma' \circ \pi(g)(\kappa')(b)\end{aligned}$$

where the last equality holds by Lemma 1.37. Hence, we can let $f : [\kappa']^{|b|+1} \to N'$ be defined by

$$f(u) = \pi(g)(u^{b\cup\{\kappa'\}}_{\{\kappa'\}})(u^{b\cup\{\kappa'\}}_b)$$

and $a = b \cup \{\kappa'\}$. Note that $\mu < \tau$ as $\tau = \nu$ is the largest cardinal of M and μ is a limit cardinal in M because G is weakly amenable. Hence, $\mathrm{lh}(G) < \pi(\mu) < \pi(\tau) = \tau'$ so indeed $a \in [\tau']$. Moreover,

$$\sigma'(f)(a) = \sigma' \circ \pi(g)(a) = \sigma' \circ \pi(g)(\kappa')(b) = \xi$$

So, f and a are as desired. This proves that the only generator of F' is κ' and thus $\nu' = \nu(F')$.

We now check that F' is weakly amenable. As we mentioned in the previous paragraph, $\mu \neq \tau$ because G is weakly amenable with respect to to M. It follows that π maps τ cofinally to τ'. As F is weakly amenable $\mathrm{dom}(F) = \mathcal{P}(\kappa) \cap N$, so $\pi[\mathrm{dom}(F)] \subseteq \mathrm{dom}(F')$ will be cofinal in $J^{E'}_{\tau'}$. Hence, $\mathrm{dom}(F') = \mathcal{P}(\kappa') \cap N'$ so F' is weakly amenable.

Letting α' and $E_{\alpha'}$ be defined as in the previous case, it follows as above that $M' = (J^{E'}_{\alpha'}, E_{\alpha'})$ is a type I potential premouse. Furthermore, in the same manner as Case 1 we can check that proper initial segments of M' are sound because π is Σ_1-preserving. This completes the verification that N' is the expansion a type I premouse in this case. $\square$ Case 2

In either case, N' is the expansion of the type I premouse $M' = (J^{E'}_{\alpha'}, E_{\alpha'})$ which completes the proof. $\square$

Lemma 1.39. *Let M be a type II premouse and $N = (J^E_\beta, F)$ the expansion of M. Assume G is a weakly amenable extender with respect to M and $\mu = \mathrm{cr}(G) < \rho^k_M$ (in particular $\mu < \alpha$). Then, $N' = \mathrm{Ult}^k(N, G)$ is the expansion of a type II premouse.*

Proof. Let us retain all the notation from the previous lemma. The proof of this lemma is similar to the previous one.

Case 1: Assume $k > 0$.

As we remarked above, being a generator of the top extender of a coherent structure is a Π_1-statement and being the only generator is a Π_2-statment. Similarly, being the largest generator of the top extender is a Π_2-statement. Hence, if γ is the largest generator of F, then because π is Σ_2-preserving $\pi(\gamma)$ will be the largest generator of F'. It follows that N' is type II. Moreover, because M satisfies the initial segment condition $F \restriction \gamma \in N$, so $F' \restriction \pi(\gamma) = \pi(F \restriction \gamma) \in N'$. So if $\xi < \nu'$ is a generator of F' then $F' \restriction \xi = (F' \restriction \pi(\gamma)) \restriction \xi \in N'$. Thus, N' satisfies the initial segment condition.

The rest of the argument as in Case 1 of Lemma 1.38 shows that $M' = (J^{E'}_{\alpha'}, E_{\alpha'})$ where $\alpha' = \nu'^{+N'}$ is a type II premouse. $\square$ Case 1

Case 2: Assume $k = 0$.

The argument in Case 2 of Lemma 1.38 works here with some slight modification. The main difference is in verifying that N' is type II and satisfies the initial segment condition. Let γ be the largest generator of F. It follows that $\pi(\gamma)$ is a generator of F' as π is Σ_1-preserving. So to verify N' is type II, we need to show there are no generators of F' above $\pi(\gamma)$. Similar to Equation 1.3, we show that if $\xi < \lambda'$, then

$$\xi = \sigma'(f)(a) \qquad \text{for some } a \in [\pi(\gamma)+1]^{<\omega} \text{ and } f : [\kappa']^{|a|} \to N' \text{ with } f \in N' \tag{1.4}$$

As before, there is some $h : \mu \to N$, $h \in N$ and $b \in [\mathrm{lh}(G)]^{<\omega}$ such that $\xi = \pi(h)(b)$. As N is a coherent structure, it follows that $h = \sigma(g)(c)$ for some $g : \kappa \to J^E_\tau$ and $c \in [\gamma+1]^{<\omega}$ So, we have

$$\begin{aligned} \xi = \pi(h)(b) = \pi(\sigma(g)(c))(b) &= \pi \circ \sigma(g)(\pi(c))(b) \\ &= \sigma' \circ \pi(g)(\pi(c))(b) \end{aligned}$$

where the last equality holds by Lemma 1.37. So, if we define f analogously to what we did in Lemma 1.38 and we let $a = \pi(c) \cup b$, we will have

$$\sigma'(f)(a) = \sigma' \circ \pi(g)(a) = \sigma' \circ \pi(g)(\pi(c))(b) = \xi$$

The only thing to note is that $b \in [\mathrm{lh}(G)]^{<\omega}$ and $\mathrm{lh}(G) < \pi(\mu) \leq \pi(\gamma)$. This is because $\mu < \nu = \gamma + 1$ as $\mu < \alpha$ is a limit cardinal in M and there are no cardinals between ν and α inside M.

Now, we need to verify that N' satisfies the ISC. However, the argument from Case 1 works here: $F \upharpoonright \gamma \in N$ so that $F' \upharpoonright \pi(\gamma) = \pi(F \upharpoonright \gamma) \in N'$. The verification of the other conditions for M' to be a premouse are proved in the exact same way as Lemma 1.38. $\square$ Case 2

This completes the proof that N' is the expansion of a type II premouse. $\square$

Before we head into the similar lemma for type III premice, we prove

Lemma 1.40. *Let M be a type III premouse and $N = (J^E_\beta, F)$ its expansion. Then, $\nu(F) = \rho^1_N$, N is 1-sound, and $p^1_N = \emptyset$.*

Proof. Let $\nu = \nu(F)$, $\kappa = \mathrm{cr}(F)$, $\tau = \kappa^{+N}$ and $\lambda = \mathrm{lh}(F)$. We first claim that $h_N(\nu) = N$. For this, it suffices to show $\beta \subseteq h_N(\nu)$. Let $\sigma : J^E_\tau \to_F J^E_\beta$ be the expansion ultrapower map. If $\zeta < \beta$ then $\zeta = \sigma(f)(a)$ for some $f : [\kappa]^{|a|} \to J^E_\tau \in J^E_\tau$ and $a \in [\nu]^{<\omega}$. As $J^E_\tau \subseteq h_N(\tau) \subseteq h_N(\nu)$ it follows that $f \in h_N(\nu)$. Recall, by the proof of Lemma 1.37 the statement "$y = \sigma(x)$" is $\Delta_1(N)$, so

$$N \models (\exists g)\ g = \sigma(f) \wedge \zeta = g(a)$$

So, $\zeta \in h_N(\nu)$.

Hence, $\rho^1_N \leq \nu$. We now show $\nu \leq \rho^1_N$. Let A be $\Sigma_1(N)$ in p and $\gamma < \nu$. We show $A \cap \gamma \in N$. Let $f \in J^E_\tau, b \in [\nu]^{<\omega}$ such that $p = \sigma(f)(b)$. Let γ^* be a generator of F such that $\gamma^* > \max(b), \gamma$. By the ISC, $F^* = F \upharpoonright \gamma^* \in N$. Let B be $\Sigma_0(N)$ such that

$$A(v) \longleftrightarrow (\exists z)\, B(v, z, p)$$

Let $\psi(v, z, p)$ be the Σ_0-definition of B. Because σ is cofinal, for $\zeta < \gamma$ we have

$$\begin{aligned} A(\zeta) &\longleftrightarrow (\exists x \in J^E_\tau)(\exists z \in \sigma(x))\, B(\zeta, z, p) \\ &\longleftrightarrow (\exists x \in J^E_\tau)(\exists z \in \sigma(x))\, B(\sigma(pr)(\{\zeta\}), z, \sigma(g)(b)) \text{ where } pr(\{\alpha\}) = \alpha \\ &\longleftrightarrow (\exists x \in J^E_\tau)\ \{u \in [\kappa]^{|b|+1} \mid J^E_\tau \models (\exists z \in x)\psi(pr(u_\zeta), z, g^u_b(u_b))\} \in F^*_{b \cup \{\zeta\}} \end{aligned}$$

Hence, $A \cap \gamma$ is $\mathbf{\Sigma}_0(N)$, so $A \cap \gamma \in N$. It follows that $\nu \leq \rho^1_N$.

We have shown that $\nu = \rho^1_N$, and the rest of the lemma follows from the fact that $N = h_N(\nu)$ which we already established. $\square$

Lemma 1.41. *Let M be a type III premouse and $N = (J^E_\beta, F)$ the expansion of M. Assume G is a weakly amenable extender with respect to M, $\mu = \mathrm{cr}(G) < \rho^k_M$, and $\mu < \nu(E^M_{top})$. Then, $N' = \mathrm{Ult}^k(N, G)$ is the expansion of a type III premouse.*

Remark 1.42. Note that for type III premouse, we need to assume that $\mathrm{cr}(G) < \nu(E^M_{\mathrm{top}})$. We did not have to assume this for type I and type II premice because the weak amenability of

G already implied this was the case. In what follows this will not be a problem as we are most concerned with normal iterations of premice. The rules of normal iterations guarantee that if G is an extender used in the iteration and applied to M, then $\mathrm{cr}(G) < \nu(E^M_{\mathrm{top}})$.

Proof. Again, we will use the same notation as the previous lemmas for type I and type II premice. However, we will in general not obtain the equality $\nu' = \pi(\nu)$. This proof is similar, but requires a bit more argument as M is type III. Let us note that in this case $\rho^1_N = \nu$ as proved in the previous lemma.

Case 1: Assume $k > 1$.
In this case, π is $\Sigma^{(1)}_2$-preserving, and $\pi(\rho^1_N) = \rho^1_{N'}$. First, note that

$$N \models (\forall \xi)\ \xi \text{ is a generator of } F \longrightarrow \xi < \nu$$

This is a Π_2-statement so, it is preserved by π. Hence, every generator of F' is less than $\pi(\nu) = \nu'$. Moreover, as $\nu = \rho^1_N$,

$$N \models (\forall \xi^1_1)(\exists \xi^1_2 > \xi^1_1)\ \xi^1_2 \text{ is a generator of } F$$

This is a $\Pi^{(1)}_2$-statement, so again π preserves it. Hence, there are unboundedly many generators of F' below ν' and it follows that N' is type III and $\nu' = \nu(F')$. Similarly, because N satisfies the initial segment condition

$$N \models (\forall \xi^1)(\exists x^0)\ x^0 = F \restriction \xi^1$$

Note, that $x^0 = F \restriction \xi^1$ is Σ_1 so the whole statement is $\Pi^{(1)}_1$ and preserved by π. Hence, N' satisfies the initial segment condition as well.

In the same manner as Case I of lemma 1.38, we can show that $M' = (J^{E'}_{\alpha'}, E_{\alpha'})$ satisfies all the conditions to be a premouse, so N' is the expansion of a type III premouse. We remark that in this case, we do get the equality $\pi(\nu) = \nu(F')$. $\square$ Case 1

Case 2: Assume $k = 0$.

In this case, we may not have the equality $\pi(\nu) = \nu(F')$. Instead, we will show $\nu(F') = \sup \pi[\nu]$, so we set $\nu' = \sup \pi[\nu]$ and note that $\alpha' = \nu'^{+N'}$ is defined in terms of this newly defined ν'.

This is similar to Case 2 of the previous lemmas. The main thing to show is that F' is type III and that N' satisfies the initial segment condition. If $\xi < \nu$ is a generator of F then because π is Σ_1-preserving, $\pi(\xi)$ is a generator of F'. So, our goal is to show there are no generators of F' larger than $\nu' = \sup \pi[\nu]$. Similar, to the previous lemmas if $\xi < \lambda'$ we show

$$\xi = \sigma'(f)(a) \qquad \text{for some } a \in [\nu']^{<\omega} \text{ and } f : [\kappa']^{|a|} \to N' \text{ with } f \in N' \tag{1.5}$$

With the same reasoning as before, we can find $h \in N$, $b \in [\mathrm{lh}(G)]^{<\omega}$, $g \in J^E_\tau$, and $c \in [\nu]^{<\omega}$ such that

$$\begin{aligned}\xi = \pi(h)(b) = \pi(\sigma(g)(c))(b) &= \pi \circ \sigma(g)(\pi(c))(b) \\ &= \sigma' \circ \pi(g)(\pi(c))(b)\end{aligned}$$

We then define f similarly to before and set $a = \pi(c) \cup b$ to obtain $\sigma'(f)(a) = \xi$. We remark that indeed $b \in [\nu']^{<\omega}$, because we assumed $\mu < \nu$.

Because we know that all generator of F' are below $\nu' = \sup \pi[\nu]$ verifying the initial segment condition is simple since for cofinally many $\xi < \nu$, $F \restriction \xi \in N$ so that $F' \restriction \pi(\xi) = \pi(F \restriction \xi) \in N'$. Using the arguments from the previous lemmas we can show that M' satisfies all the conditions of a premouse, so N' is the expansion of a type III premouse. $\square$ Case 2

Case 3: Assume $k = 1$.

In this case, we show that actually $N' = \mathrm{Ult}^0(N, G)$ and then we can just argue as in the previous case. First, note that $\sigma : J_\tau^E \to J_\beta^E$ and $\sigma' : J_{\tau'}^{E'} \to J_{\beta'}^{E'}$ are both cofinal. We claim that $\pi : J_\beta^E \to J_{\beta'}^{E'}$ is also cofinal. To see this, first note that $\sup \pi[\tau] = \tau'$. This is because $\mu \neq \tau$ as μ is a limit cardinal in N and $\tau < \rho_N^1$. So, any function good $\mathbf{\Sigma}_1^{(0)}(N)$-function $f : \mu \to \tau$ is an element of N. Now, to see that π is cofinal, fix $\zeta < \beta'$. Because σ' is cofinal, there is $\xi_0 < \tau'$ such that $\sigma'(\xi_0) > \zeta$. Moreover, because π maps τ cofinally to τ' there is $\xi < \tau$ such that $\pi(\xi) > \xi_0$ so $\sigma' \circ \pi(\xi) > \zeta$. Then, by Lemma 1.37,

$$\zeta < \sigma' \circ \pi(\xi) = \pi \circ \sigma(\xi)$$

So, $\pi : J_\beta^E \to J_{\beta'}^{E'}$ is cofinal.

Now, given any $x \in N'$ let f be a good $\Sigma_1^{(0)}(N)$-function in p and $a \in [\mathrm{lh}(G)]^{<\omega}$ such that $\pi(f)(a) = x$. We show that we can replace f with a function $g \in N$. Let $F(u, v, s)$ be the Σ_1 functionally absolute defintion of f, such that

$$f(x) = y \longleftrightarrow F(x, y, p)$$

Suppose $F(u, v, s) \equiv (\exists z)\, \bar{F}(u, v, s, z)$ where $\bar{F}$ is $\Sigma_0(N)$. Letting $\bar{F}'$ be $\Sigma_0(N')$ by the same definition as $\bar{F}$, fix $z \in N'$ witnessing $x = \pi(f)(a)$, i.e. $\bar{F}'(x, a, z, \pi(p))$. Because π is cofinal, $z \in \pi(w)$ for some $w \in N$. So, define the function g as follows

$$g(u) = \begin{cases} v \text{ if } v \text{ exists such that } (\exists z \in w)\, \bar{F}(u, v, p, z) \\ 0 \text{ otherwise} \end{cases}$$

Then, $g \in N$, and $\pi(g)(a) = \pi(f)(a) = x$. Hence, $N' = \mathrm{Ult}^0(N, G)$ and then arguing as in the previous case, we get that N' is the expansion of a type III premouse. $\square$ Case 3

$\square$

Remark 1.43. Continuing with the notation of the previous lemma, let us remind that we showed if $N' = \mathrm{Ult}^k(N, G)$ for $k > 1$, then $\pi(\nu) = \nu(F')$. Otherwise, then $N' = \mathrm{Ult}^0(N, G)$ and $\nu(F') = \sup \pi[\nu]$ (and it is possible $\pi(\nu) > \nu(F')$). However, in either case $\pi(\nu) \geq \nu(F')$.

1.4 Iterations and Phalanxes

Definition 1.44. A *iteration tree order* on an ordinal α is a strict partial order $<_T$ such that

- $\beta <_T \gamma \implies \beta < \gamma$
- $\{\beta \mid \beta <_T \gamma\}$ is well ordered
- γ is a successor ordinal iff γ is a T-successor
- if γ is limit then $\gamma = \sup\{\beta \mid \beta <_T \gamma\}$

Definition 1.45. An *iteration $\mathcal{T}$ of length* θ of a premouse M consists of

- an iteration tree ordering T on θ
- premice $M_i^{\mathcal{T}}$ for $i < \theta$
- commutative partial maps $\pi_{i,j}^{\mathcal{T}} : M_i^{\mathcal{T}} \to M_j^{\mathcal{T}}$ for $i \leq_T j$
- for $i + 1 < \theta$ setting $\xi_i = T(i + 1)$, we have $\eta_i \leq \mathrm{ht}(M_{\xi_i}^{\mathcal{T}})$ and for every $j < \theta$ there are only finitely many i such that $\xi_i <_T j$ and $\eta_i < \mathrm{ht}(M_{\xi_i}^{\mathcal{T}})$
- a set *padding set* $B \subseteq \theta$ such that if $i \notin B$, then $i + 1 < \theta$ and $E_i^{\mathcal{T}}$ is an extender from the $M_i^{\mathcal{T}}$-sequence . Moreover, setting $\kappa_i = \mathrm{cr}(E_i^{\mathcal{T}})$, $\alpha_i = \mathrm{lh}(E_i^{\mathcal{T}})$, and $\tau_i = \kappa_i^{+M_i^{\mathcal{T}}||\alpha_i}$ we have $\tau_i = \kappa_i^{+M_{\xi_i}^{\mathcal{T}}||\eta_i}$, $M_{\xi_i}^{\mathcal{T}}|\tau_i = M_i^{\mathcal{T}}|\tau_i$ and

$$M_{i+1}^{\mathcal{T}} = \mathrm{Ult}^*(M_{\xi_i}^{\mathcal{T}}||\eta_i, E_i^{\mathcal{T}}) \text{ and } \pi_{\xi_i,i+1}^{\mathcal{T}} : M_{\xi_i}^{\mathcal{T}}||\eta_i \to M_{i+1}^{\mathcal{T}} \text{ is the ultrapower map}$$

If $i \in B$ and $i+1 < \theta$ then $i = T(i+1)$, $M^{\mathcal{T}}_{i+1} = M^{\mathcal{T}}_i$ and $\pi^{\mathcal{T}}_{i,i+1} = \mathrm{id}$

- if $i < \theta$ is limit, then $M^{\mathcal{T}}_i$ is the direct limit of the system $(M^{\mathcal{T}}_k, \pi^{\mathcal{T}}_{\bar{k},k} \mid \bar{k} \leq_T k <_T i)$ and $\pi^{\mathcal{T}}_k : M^{\mathcal{T}}_k \to M^{\mathcal{T}}_i$ are the direct limit maps for $k <_T i$

Notation 1.46. If $\mathcal{T}$ is a an iteration of the premouse M, then we often will drop the superscript $\mathcal{T}$ whenever possible. The most pertinent information from an iteration will be summarized as:

- $\kappa_i = \mathrm{cr}(E_i)$, $\alpha_i = \mathrm{lh}(E_i)$, $\nu_i = \nu(E_i)$, $\tau_i = \kappa_i^{+M_i||\alpha_i}$
- $\xi_i = T(i+1)$, $M^*_{i+1} = M_{\xi_i}||\eta_i$, $\mathrm{lh}(\mathcal{T}) = \theta$

Definition 1.47. An iteration $\mathcal{T}$ of M is *normal* iff

- $\alpha_i < \alpha_j$ whenever $i, j \notin B$ and $i < j$
- $\xi_i =$ the least ξ such that $\kappa_i < \nu_\xi$
- $\eta_i =$ the largest $\eta \leq \mathrm{ht}(M_{\xi_i})$ such that $\tau_i = \kappa_i^{+M_{\xi_i}||\eta}$

Definition 1.48. An *iteration strategy* Σ for a premouse M, is a partial function defined on iterations $\mathcal{T}$ of M with limit length such that if $b = \Sigma(\mathcal{T})$ then b is a cofinal well-founded branch through $\mathcal{T}$.

Definition 1.49. An iteration $\mathcal{T}$ of a premouse M is a Σ-*iteration* (or *by* Σ) iff for any limit $\lambda < \mathrm{lh}(\mathcal{T})$ $\{i \mid i <_T \lambda\} = \Sigma(\mathcal{T} \restriction \lambda)$

Definition 1.50. We say Σ is a *normal iteration strategy* for a premouse M iff any normal iteration $\mathcal{T}$ by Σ can be continued.

To compare two iterable premice M and N, we can follow the usual method of iterating by least-disagreement. However, in many cases one would like to guarantee that the critical points along the main branch of one side of this coiteration do not drop below a predetermined point. Often, it is not possible to guarantee this in the comparison between M and N, so we often work with structures known as *phalanxes*.

Definition 1.51. The triple (M_{-1}, M_0, ν) is a *phalanx* iff

- Each M_i is a premouse
- $\nu \in M_{-1} \cap M_0$ and is a cardinal in M_0
- $M_{-1}|\nu = M_0|\nu$

Definition 1.52. A *normal iteration* iteration of a phalanx (M_{-1}, M_0, ν) is defined similarly as for a premouse, but we set $\nu_{-1} = \nu$ so if $\mathrm{cr}(E_i^{\mathcal{T}}) < \nu$ for some $i < \mathrm{lh}(\mathcal{T})$, then we apply it to M_{-1}.

We are leaving out a technical wrinkle in the above definition. In the iteration, say $\mathcal{T}$, of a phalanx (M_{-1}, M_0, ν), it is possible that for some $i < \mathrm{lh}(\mathcal{T}), \mathrm{cr}(E_i^{\mathcal{T}})$ is the largest cardinal below ν in M_{-1}. If this is the case, then when forming $M_{i+1}^{\mathcal{T}}$, $E_i^{\mathcal{T}}$ stretches the top extender of $M_{i+1}^{*\mathcal{T}}$. It follows that $M_{i+1}^{\mathcal{T}}$ does not satisfy the ISC, so is not actually a premouse. Moreover, $\mathrm{ht}(M_{i+1}^{\mathcal{T}}) = \alpha_i$, so at stage $i+1$ the only possible extender to pick (to "preserve" normality) is the top extender of $M_{i+1}^{\mathcal{T}}$, but then $\alpha_i = \alpha_{i+1}$. We then apply this top extender to an initial segment of M_{-1} and the resulting ultrapower $M_{i+2}^{\mathcal{T}}$ will be a premouse. Additionally, we do not apply any extenders to $M_{i+1}^{\mathcal{T}}$, as $\nu_{i+1}^{\mathcal{T}} = \nu_i^{\mathcal{T}}$, so $\{0, i+1\}$ is a maximal branch in $\mathcal{T}$.

Regarding our remark preceding the defintion of phalanx, the ordinal ν in the phalanx (M_{-1}, M_0, ν) will play the role of "the predetermined point" that we do not want critical points to drop below and will allow us to complete comparison arguments.

Definition 1.53. A *coiteration* of a premouse M with a premouse N is a pair of normal iterations $(\bar{\mathcal{T}}, \mathcal{T})$ such that $\bar{\mathcal{T}}$ is a normal iteration of M and $\mathcal{T}$ is a normal iteration of the premouse N, and the extenders in the iteration are chosen by least disagreement. Similarly, we define a *coiteration* of phalanx (M_{-1}, M_0, ν) with a premouse N, but in this case $\bar{\mathcal{T}}$ is a normal iteration of (M_{-1}, M_0, ν).

We now list some basic facts about iterations (or coiterations) of premice and phalanxes. The proofs of all of these can be found in [16].

Fact 1.54. *(1) In a normal iteration $\mathcal{T}$ of a premouse M, each extender $E_i^{\mathcal{T}}$ is close to $M_{i+1}^{*\mathcal{T}}$. The similar statement holds in an iteration of a phalanx, except when $i+1$ is an anomaly.*

(2) In a normal iteration $\mathcal{T}$ of a premouse $E_i^{\mathcal{T}}$ is incompatible with $E_j^{\mathcal{T}}$ for any $i < j$. The similar statement holds in an iteration of a phalanx.

(3) In a normal iteration $\mathcal{T}$ of a premouse M, if the iteration map $\pi_{\xi_i,j}^{\mathcal{T}} : M_{i+1}^{*\mathcal{T}} \to M_j^{\mathcal{T}}$ is total then it is Σ^*-preserving and $M_{i+1}^{*\mathcal{T}}$ and $M_j^{\mathcal{T}}$ share projecta below κ_i. The similar statement holds in a normal iteration of a phalanx.

(4) In the coiteration $(\bar{\mathcal{T}}, \mathcal{T})$ of two premice M, N, for any i and j the extender $E_i^{\bar{\mathcal{T}}}$ is incompatible with the extender $E_j^{\mathcal{T}}$. The similar statement holds in the coiteration $(\bar{\mathcal{T}}, \mathcal{T})$ of a phalanx (M_{-1}, M_0, ν) and a premouse N except possibly when i is an anomaly.

(5) The coiteration $(\bar{\mathcal{T}}, \mathcal{T})$ of two premice M, N, terminates before $\max\{\mathrm{card}(M)^+, \mathrm{card}(N)^+\}$. *The similar statement holds for the coiteration $(\bar{\mathcal{T}}, \mathcal{T})$ of a phalanx (M_{-1}, M_0, ν) and a premouse N.*

1.5 Copy Constructions and Condensation

Before heading into the copy construction we recall the shift lemma.

Lemma 1.55 (Shift Lemma). *Assume $\bar{M}, M$ are premice of the same type and $\sigma : \bar{M} \to M$ is a premouse embedding. Let $\bar{\nu} = \nu(E_{top}^{\bar{M}})$ and $\nu = \nu(E_{top}^{M})$. Suppose $\bar{G}$ is a weakly amenable $(\bar{\mu}, \bar{\lambda})$-extender with respect to $\bar{M}$ with $\bar{\mu} < \bar{\nu}$ and G is a weakly amenable (μ, λ)-extender with respect to M where $\mu = \sigma(\bar{\mu}) < \nu$. Suppose further there is an order-preserving function $k : \bar{\lambda} \to \lambda$ such that for all $a \in [\bar{\lambda}]^{<\omega}$ and $x \in \mathcal{P}([\bar{\mu}]^{|a|}) \cap \bar{M}$ if $x \in \bar{G}_a$ then $\sigma(x) \in G_{k[a]}$. Suppose σ is $\Sigma_0^{(n)}$-preserving for n such that $\rho_{\bar{M}}^{n+1} \leq \bar{\mu} < \rho_{\bar{M}}^{n}$. If $M' = \mathrm{Ult}^*(M, G)$ exists then so does $\bar{M}' = \mathrm{Ult}^*(\bar{M}, \bar{G})$ and there is a $\Sigma_0^{(n)}$-preserving map $\sigma' : \bar{M}' \to M'$ such that $\sigma' \upharpoonright \bar{\lambda} = k$ and $\sigma' \circ \bar{\pi} = \pi \circ \sigma$ where $\bar{\pi} : \bar{M} \to_{\bar{G}} \bar{M}'$ and $\pi : M \to_G M'$.*

Proof. To have commutativity hold, σ' must be defined as

$$\sigma'(\bar{\pi}(f)(a)) = \pi(\sigma(f))(k[a])$$

where for good $\Sigma_1^{(n-1)}(\bar{M})$-functions f in p, $\sigma(f)$ is the good $\Sigma_1^{(n-1)}(M)$-function defined by the same functionally absolute definition as f in $\sigma(p)$. By $\Sigma_0^{(n)}$ Łoś's theorem, σ' will be $\Sigma_0^{(n)}$-preserving (at the n-th level, the weak amenability of $\bar{G}$ implies $\bar{\pi}$ is $\Sigma_0^{(n)}$-cofinal). The fact that $\sigma' \upharpoonright \bar{\lambda} = k$ is a result of the fact that ordinals $\alpha < \bar{\lambda}$ are represented by $\bar{\pi}(pr)(\{\alpha\})$ in ultapower where $pr(\{\xi\}) = \xi$. $\square$

One of the next things we would like to verify is that if we have a normal iteration tree $\bar{\mathcal{T}}$ on a premouse $\bar{M}$ and an embedding $\sigma : \bar{M} \to M$, then when we copy the tree to M, the resulting tree is normal. The next lemma will help with the verification of this.

Lemma 1.56. *Continuing with the notation of the previous lemma, let $\nu' = \nu(E_{top}^{M'})$ and $\bar{\nu}' = \nu(E_{top}^{\bar{M}'})$.*

(a) If $\bar{M}$ and M are type I premice, then $\sigma(\bar{\nu}') = \nu'$.

(b) If $\bar{M}$ and M are type II premice, then $\sigma'(\bar{\nu}') = \nu'$.

(c) If $\bar{M}$ and M are type III premice and $\sigma(\bar{\nu}) \geq \nu$, then $\sigma'(\bar{\nu}') \geq \nu'$.

Remark 1.57. Here and in the copy construction, if $\bar{M}$ and M are type III premice, $\bar{M}' = \mathrm{Ult}^0(\bar{M}, \bar{G})$, and there is a function $f : \bar{\mu} \to \bar{\nu} \in \bar{M}$ which is cofinal, then we will set $M' = \mathrm{Ult}^1(M, G)$ i.e. we take a 1-ultrapower. Equivalently, $M' = \mathrm{Ult}^0(M, G)$ by Remark 1.43.

Proof. Let $\bar{F}$ denote the top extender of $\bar{M}$ and $\bar{\kappa} = \mathrm{cr}(\bar{F})$. Similarly, let F denote the top extender of M and $\kappa = \mathrm{cr}(F)$.

(a) If $\bar{M}$ and M are type I, then $\bar{\nu} = \bar{\kappa}^{+\bar{M}}$ and $\nu = \kappa^{+M}$. Recall that our map σ is literally defined on the expansions $\widehat{\bar{M}}$ and $\widehat{M}$. Let $\bar{\lambda}$ be the length of the top extender of $\widehat{\bar{M}}$ and λ be the length of the top extender of $\widehat{M}$. Then, $\sigma(\bar{\kappa}) = \kappa$ and $\sigma(\bar{\lambda}) = \lambda$. Inside $\widehat{\bar{M}}$, $\bar{\nu}$ is the cardinal successor of $\bar{\kappa}$. Similarly, ν is the cardinal successor of κ inside $\widehat{M}$. By considering the restriction of σ to $\widehat{\bar{M}}|\bar{\lambda}$ (which will be fully elementary), we get $\sigma(\bar{\nu}) = \nu$. By Lemma 1.38, we know that $\bar{\pi}(\bar{\nu}) = \bar{\nu}'$ and $\pi(\nu) = \nu'$. Hence,

$$\sigma'(\bar{\nu}') = \sigma' \circ \bar{\pi}(\bar{\nu}) = \pi \circ \sigma(\bar{\nu}) = \pi(\nu) = \nu'$$

(b) Recall that in the case of type II premice, we have an extra constant symbol for the top extender cutback at the largest generator. Let $\bar{\gamma} = \bar{\nu} - 1$ and $\gamma = \nu - 1$ be the largest generators of $\bar{F}$ and F respectively. Because of the additional constant symbol, it follows that $\sigma(\bar{F} \restriction \bar{\gamma}) = F \restriction \gamma$. We claim that $\sigma(\bar{\gamma}) = \gamma$, and hence $\sigma(\bar{\nu}) = \nu$. First, note that $(\bar{\kappa}, \bar{\gamma}) \in \bar{F} \restriction \bar{\gamma}$ so $(\sigma(\bar{\kappa}), \sigma(\bar{\gamma})) = (\kappa, \sigma(\bar{\gamma})) \in F \restriction \gamma$. However, γ is the unique y such that $(\kappa, y) \in F \restriction \gamma$. Hence, $\sigma(\bar{\gamma}) = \gamma$. By Lemma 1.39, we know $\bar{\pi}(\bar{\nu}) = \bar{\nu}'$ and $\pi(\nu) = \nu'$ and it follows that

$$\sigma'(\bar{\nu}') = \sigma' \circ \bar{\pi}(\bar{\nu}) = \pi \circ \sigma(\bar{\nu}) = \pi(\nu) = \nu'$$

(c) Because $\bar{M}$ and M are type III, $\bar{\nu} = \rho^1_{\bar{M}}$ and $\nu = \rho^1_M$. Let us first remark that the condition $\sigma(\bar{\nu}) \geq \nu$ happens in many cases. For example, if σ is $\Sigma_0^{(1)}$-preserving and $\sigma^{-1}[\rho^1_M] \subseteq \rho^1_{\bar{M}}$. Or if σ is $\Sigma_1^{(1)}$-preserving, because then $\sigma^{-1}[\rho^1_M] \subseteq \rho^1_{\bar{M}}$ as can be seen by considering the formula $(\exists \xi^1)\ v = \xi^1$ in M. In either of these cases, $\sigma(\bar{\nu}) \geq \nu$.

As mentioned in Remark 1.43, there are two cases to consider. Either $\bar{M}' = \mathrm{Ult}^n(\bar{M}, \bar{G})$ for $n > 1$ and $\pi(\bar{\nu}) = \bar{\nu}'$ or $\bar{M}' = \mathrm{Ult}^0(\bar{M}, \bar{G})$ and $\bar{\nu}' = \sup \bar{\pi}[\bar{\nu}]$.

Let's first consider the case where $\bar{M}' = \mathrm{Ult}^n(\bar{M}, \bar{G})$ for $n > 1$ and $\bar{\pi}(\bar{\nu}) = \bar{\nu}'$. It follows that $M' = \mathrm{Ult}^k(M, G)$ for $k \geq n > 1$ so $\pi(\nu) = \nu'$. We then have

$$\sigma'(\bar{\nu}') = \sigma' \circ \bar{\pi}(\bar{\nu}) = \pi \circ \sigma(\bar{\nu}) \geq \pi(\nu) = \nu'$$

So, $\sigma'(\bar{\nu}') \geq \nu'$ in this case.

Now assume that $\bar{M}' = \mathrm{Ult}^0(\bar{M}, \bar{G})$ and $\bar{\nu}' = \sup \pi[\bar{\nu}]$. It is possible $M' = \mathrm{Ult}^n(M, G)$ for $n > 1$ in which case $\pi(\nu) = \nu'$. Otherwise, $M' = \mathrm{Ult}^0(M, G)$ and $\nu' = \sup \pi[\nu]$. However, in either case $\pi(\nu) \geq \nu'$. We claim that $\sigma'(\bar{\nu}') \geq \nu'$. Suppose this was not the case. Then, $\sigma'(\bar{\nu}') < \nu'$ and because $\sigma(\bar{\nu}) \geq \nu$ we have

$$\sigma'(\bar{\nu}') < \nu' \leq \pi(\nu) \leq \pi \circ \sigma(\bar{\nu}) = \sigma' \circ \bar{\pi}(\bar{\nu})$$

So, $\bar{\nu}' < \bar{\pi}(\bar{\nu})$. Hence, there is an $\bar{f} : \bar{\mu} \to \bar{\nu} \in \bar{M}$ which is cofinal and without loss of generality, increasing as well. By Remark 1.57, this means $M' = \mathrm{Ult}^1(M, G)$, so $\nu' = \sup \pi[\nu]$. Notice then that $\sup \pi(\bar{f})[\bar{\mu}] = \sup \pi[\bar{\nu}] = \bar{\nu}'$, so

$$\bar{M}' \models (\forall \zeta < \bar{\mu})\ \bar{\pi}(\bar{f})(\zeta) < \bar{\nu}' \tag{1.6}$$

Now, consider $f = \sigma(\bar{f})$. By elementarity of σ, $f : \mu \to \sigma(\bar{\nu})$ is cofinal and increasing.

It follows that

$$\sup \pi(f)[\mu] = \sup \pi[\sigma(\bar{\nu})] \geq \sup \pi[\nu] = \nu' \tag{1.7}$$

Note that $\pi(f) = \sigma' \circ \bar{\pi}(\bar{f})$ and $\sigma'(\bar{\mu}) = \mu$. Applying the elementarity of σ' to Equation 1.6 gives

$$M' \models (\forall \zeta < \mu)\ \pi(f)(\zeta) < \sigma'(\bar{\nu}')$$

This says $\sup \pi(f)[\mu] \leq \sigma'(\bar{\nu}') < \nu'$ contradicting Equation 1.7. Hence, $\sigma'(\bar{\nu}') \geq \nu'$ as claimed.

$\square$

Next, we review the copy construction.

Definition 1.58. Let $\bar{M}$ and M be premice and $\sigma : \bar{M} \to M$ be Σ_0-preserving. Suppose $\bar{\mathcal{T}}$ is a normal iteration of $\bar{M}$. The *copy* of $\bar{\mathcal{T}}$ via σ is the iteration $\mathcal{T} = \sigma\bar{\mathcal{T}}$ of M with copying maps σ_i for $i < \mathrm{lh}(\bar{\mathcal{T}})$ satisfying

- The iterations have the same length θ, same tree structure T, and same padding set B
- The maps $\sigma_i : M_i^{\bar{\mathcal{T}}} \to M_i^{\mathcal{T}}$ commute with iteration maps i.e. $\sigma_i \circ \pi_{i,h}^{\bar{\mathcal{T}}} = \pi_{i,h}^{\mathcal{T}} \circ \sigma_h$ for $h \leq_T i$.
- For all $h < i < \theta$, $\sigma_i \upharpoonright \alpha_h^{\bar{\mathcal{T}}} = \sigma_h \upharpoonright \alpha_h^{\bar{\mathcal{T}}}$
- For all $h + 1 < \theta$, $\eta_h^{\mathcal{T}} = \sigma_h(\eta_h^{\bar{\mathcal{T}}})$
- If $h + 1 < \theta$ and $h \in B$, then $\sigma_{h+1} = \sigma_h \upharpoonright (M_h^{\bar{\mathcal{T}}} || \eta_h^{\bar{\mathcal{T}}})$
- If $h+1 < \theta$ and $h \notin B$, then letting $\sigma^* = \sigma_{\xi_h} \upharpoonright M_{h+1}^{*\bar{\mathcal{T}}} : M_{h+1}^{*\bar{\mathcal{T}}} \to M_{h+1}^{*\mathcal{T}}$ and $k = \sigma_h \upharpoonright \alpha_h^{\bar{\mathcal{T}}}$, σ_{h+1} is obtained from the shift lemma using σ^* and k

- If $i < \theta$ is limit, then $\sigma_i(x) = \pi^{\mathcal{T}}_{h,i} \circ \sigma_h(\bar{x})$ whenever $h <_T i$ and $\bar{x} \in M^{\bar{\mathcal{T}}}_h$ is such that $x = \pi^{\bar{\mathcal{T}}}_{h,i}(\bar{x})$

Definition 1.59. Let $\sigma : \bar{M} \to M$ be phalanx embedding, and assume M has an iteration strategy Σ. The *copied strategy* $\sigma\Sigma$ is an iteration strategy for $\bar{M}$ and defined so that if $\bar{\mathcal{T}}$ is an iteration of limit on $\bar{M}$, then $\sigma\Sigma(\bar{\mathcal{T}}) = \Sigma(\sigma\bar{\mathcal{T}})$.

Lemma 1.60. *Let $\sigma : \bar{M} \to M$ be a cardinal and $\Sigma_0^{(n)}$-preserving embedding of premice. In the case that $\bar{M}$ and M are type III, assume further that $\sigma(\rho^1_{\bar{M}}) \geq \rho^1_M$ if $n = 1$. Let Σ be a normal iteration strategy for M and $\bar{\mathcal{T}}$ a normal iteration of $\bar{M}$ above $\rho^{n+1}_{\bar{M}}$ using the copied strategy $\sigma\Sigma$. Then, $\mathcal{T} = \sigma\bar{\mathcal{T}}$ exists and is a normal Σ-iteration of M.*

Proof. This is proved in [16] and most of the proof follows in our setting. We index our extenders differently then those found in [16], so the main thing to verify in our case is that the copied tree structure is indeed normal. This amounts to checking that for $i+1 < \mathrm{lh}(\mathcal{T})$, if $\xi = T(i+1)$, then ξ is the least ζ such that $\kappa^{\mathcal{T}}_i < \nu^{\mathcal{T}}_\zeta$.

Suppose $\theta = \mathrm{lh}(\bar{\mathcal{T}})$. Let $\bar{M}_i, \bar{E}_i, \bar{\kappa}_i, \bar{\nu}_i, \bar{\xi}_i$ denote the models, extenders, and iteration data from $\bar{\mathcal{T}}$ and $M_i, E_i, \kappa_i, \nu_i$ denote the models, extenders, and iteration data from $\mathcal{T}$. Let $\sigma_i : \bar{M}_i \to M_i$ denote the copy maps. We show that for all $i + 1 < \theta$

$$\nu_j \leq \kappa_i \text{ for all } j < \bar{\xi}_i \text{ and } \kappa_i < \nu_{\bar{\xi}_i} \tag{1.8}$$

For type I or type II premice, this is easily verified as for all $i < \theta$, $\sigma_i(\bar{\nu}_i) = \nu_i$. This is true trivially if $\bar{E}_i$ is not the top extender of $\bar{M}_i$, and otherwise it holds by Lemma 1.56. By the normality of $\bar{\mathcal{T}}$, we have $\bar{\nu}_j \leq \bar{\kappa}_i$ for $j < \bar{\xi}_i$ and $\bar{\kappa}_i < \bar{\nu}_{\xi_i}$. The agreement in the copy construction guarantees $\sigma_{\xi_i} \restriction \bar{\alpha}_j = \sigma_j \restriction \bar{\alpha}_j$ for $j < \xi_i$ and so it follows that $\nu_j \leq \kappa_i$ for $j < \bar{\xi}_i$ and $\kappa_i < \nu_{\xi_i}$.

Now, suppose $\bar{M}$ and M are type III premice. Let us note that in the case of type III premice, we would always start with a $\Sigma_0^{(1)}$-preserving map as the first projectum is the natural length of the top extender. The copy construction then guarantees that each σ_i will be $\Sigma_0^{(1)}$-preserving. In this case we do not necessarily have the equality $\sigma_i(\bar{\nu}_i) = \nu_i$, but by Lemma 1.56 we do at least know $\sigma_i(\bar{\nu}_i) \geq \nu_i$. Let $\xi = \bar{\xi}_i$. Because $\bar{\mathcal{T}}$ is normal, $\bar{\kappa}_i < \bar{\nu}_\xi = \rho^1_{\bar{M}_\xi}$ and because σ_ξ is $\Sigma_0^{(1)}$-preserving, it follows that $\kappa_i < \nu_\xi = \rho^1_{M_\xi}$. We need to check that $\kappa_i \geq \nu_j$ for any $j < \xi$. Suppose this was not the case, and say $j < \xi$ is such that $\sigma_i(\bar{\kappa}_i) = \kappa_i < \nu_j$. By the agreement in the copy construction, $\kappa_i = \sigma_\xi(\bar{\kappa}_i) < \nu_j \leq \sigma_j(\bar{\nu}_j) = \sigma_\xi(\bar{\nu}_j)$. It follows that $\bar{\kappa}_i < \nu_j$ contradicting the normality of $\bar{\mathcal{T}}$. Hence, $\kappa_i \geq \nu_j$ for all $j < \xi$. $\square$

Definition 1.61. A map $\sigma : M \to N$ is an *embedding* of the phalanx (N, M, ν) into N iff $\sigma \upharpoonright \nu = \mathrm{id}$ and σ is cardinal and $\Sigma_0^{(n)}$ preserving whenever n is such that $\rho^n_M > \nu$.

Definition 1.62. Let $\bar{\mathcal{T}}$ be a normal iteration of the phalanx (N, M, ν). Let $\sigma : (N, M, \nu) \to N$ be a phalanx embedding. The *copy* of $\bar{\mathcal{T}}$ by σ is $\mathcal{T} = \sigma\bar{\mathcal{T}}$ and has *copy maps* σ_i for $-1 \leq i < \mathrm{lh}(\bar{\mathcal{T}})$ if the following conditions hold

- $\sigma_{-1} = \mathrm{id} : N \to N$, $\sigma_0 = \sigma : M \to N$ and for $i > 0$, $\sigma_i : M_i^{\bar{\mathcal{T}}} \to M_i^{\mathcal{T}}$ is defined as in the shift lemma (so they commute with the iteration maps)

- The tree structure $T = \bar{T}$ with the exception that the two roots of $\bar{T}$ are glued together

- The iteration data is preserved under the copy maps:

$$(\kappa_i^{\mathcal{T}}, \nu_i^{\mathcal{T}}, \alpha_i^{\mathcal{T}}, \tau_i^{\mathcal{T}}, \xi_i^{\mathcal{T}}, \eta_i^{\mathcal{T}}) = \sigma(\kappa_i^{\bar{\mathcal{T}}}, \nu_i^{\bar{\mathcal{T}}}, \alpha_i^{\bar{\mathcal{T}}}, \tau_i^{\bar{\mathcal{T}}}, \xi_i^{\bar{\mathcal{T}}}, \eta_i^{\bar{\mathcal{T}}})$$

Remark 1.63. If $i+1$ is an anomaly, then $i+1$ is a truncation in $\bar{\mathcal{T}}$ but not in $\mathcal{T}$. In this case, the $M_{i+1}^{\bar{\mathcal{T}}}$ is embedded into $\pi_{0,i+1}^{\mathcal{T}}(M_{i+1}^{*,\bar{\mathcal{T}}})$.

Definition 1.64. Let $\sigma : (N, M, \nu) \to N$ be phalanx embedding, and assume N has an iteration strategy Σ. The *copied strategy* $\sigma\Sigma$ is an iteration strategy for the phalanx (N, M, ν) and defined so that $\bar{\mathcal{T}}$ is an iteration of limit length for the phalanx, then $\sigma\Sigma(\bar{\mathcal{T}}) = \Sigma(\sigma\bar{\mathcal{T}})$.

Lemma 1.65. *Let $\sigma : (N, M, \nu) \to N$ be a phalanx embedding and Σ a normal iteration strategy for N. Then, the copied strategy $\sigma\Sigma$ is a normal iteration strategy for (N, M, ν). Moreover, if $\bar{\mathcal{T}}$ is a normal iteration of (N, M, ν) via $\sigma\Sigma$, then $\mathcal{T} = \sigma\bar{\mathcal{T}}$ is a normal iteration of N via Σ.*

Proof. As in Lemma 1.60 we can verify that $\mathcal{T}$ is a normal iteration and the rest is proved in [16]. $\square$

Fact 1.66. *Here we summarize some important facts about the copy construction*

(1) The copy maps satisfy $\sigma_j \upharpoonright \alpha_i^{\bar{\mathcal{T}}} = \sigma_i \upharpoonright \alpha_i^{\bar{\mathcal{T}}}$ for $i \leq j$

(2) The copy maps are cardinal preserving and σ_i is $\Sigma_0^{(n)}$-preserving where n is such that $\rho^n_{M_i^{\bar{\mathcal{T}}}} \geq \sup\{\alpha_j^{\bar{\mathcal{T}}} \mid j < i\}$. In many cases, the copy maps will be fully Σ^*-preserving, but anomalies complicate things.

To prove the solidity theorem or the condensation lemma, a version of the Dodd-Jensen lemma is needed. However, the full Dodd-Jensen lemma requires the mouse to be uniquely iterable, and in general one cannot prove the existence of a uniqueness strategy for arbitrary mice. Fortunately, using a reflection argument, we need only require a type of countable iterability and a variation of the Dodd-Jensen lemma.

Definition 1.67. Let θ, μ be ordinals. A (θ, μ)-*iteration* $\mathcal{T}$ of a premouse M is an iteration of length θ which is the linear concatenation of less than μ nomal iterations.

Definition 1.68. A premouse M is *weakly iterable* iff any countable $\bar{M}$ which elementarily embeds into M has an $(\omega_1 + 1, \omega_1)$-iteration strategy.

Theorem 1.69 (Neeman-Steel Lemma)**.** *Let M be a countable premouse with an $(\omega_1 + 1, \omega_1)$-iteration strategy. Then, M has an e-minimal $(\omega_1 + 1, \omega_1)$-iteration strategy, for any enumeration e of M.*

Remark 1.70. We say that a normal iteration strategy Σ for M is e-minimal iff whenever $\mathcal{T}$ is an iteration of M by Σ with last model M' and there is a Σ^*-preserving map $\sigma : M \to M'$, then there is no truncation on the main branch of $\mathcal{T}$ and $\pi(e) \leq_{\text{lex}} \sigma(e)$ where $\pi : M \to M'$ is the iteration map. Here, we are writing $\pi(e) \leq_{\text{lex}} \sigma(e)$ to mean $\pi(e(n)) < \sigma(e(n))$ where n is the least k such that $\pi(e(k)) \neq \sigma(e(k))$.

Using, the Neeman-Steel lemma, and the results stated above, one can prove the following useful comparison lemma

Lemma 1.71. *Let N be a countable presolid premouse and let Σ be an e-minimal (ω_1+1, ω_1)-iteration strategy for N where e is some fixed enumeration of N. Given a phalanx (N, M, ν) together with a phalanx embedding $\sigma : (N, M, \nu) \to N$, let $(\bar{\mathcal{T}}, \mathcal{T})$ be the coiteration of (N, M, ν) with N via $(\sigma\Sigma, \Sigma)$. This coiteration terminates at some countable ordinal $\theta + 1$ and letting $N_\theta = M_\theta^{\mathcal{T}}$ and $M_\theta = M_\theta^{\bar{\mathcal{T}}}$ then*

(1) $0 \leq_{\bar{T}} \theta$, i.e. M is the root model of the main branch of $\bar{\mathcal{T}}$

(2) $M_\theta \trianglelefteq N_\theta$

(3) There is no truncation point on the main branch of $\bar{\mathcal{T}}$

The Neeman-Steel Lemma is also used for

Theorem 1.72 (Solidity Theorem)**.** *Every weakly iterable premouse is solid.*

Remark 1.73. Note, that by the solidity theorem and the Neeman-Steel lemma, if N is a countable $(\omega_1 + 1, \omega_1)$-iterable premouse and there is a phalanx embedding from (N, M, ν) into N, then Lemma 1.71 holds.

The solidity theorem implies that *cores* of iterable premice exist. Cores are transitive collapses of $\Sigma_1^{(n)}$-hulls.

Definition 1.74. Let M be a premouse and $\rho^{\omega}_{M} \leq \alpha$. We say $\bar{M}$ is the *core of* M *above* α and write $\bar{M} = \text{core}_{\alpha}(M)$ iff there is a Σ^*-preserving map $\sigma : \bar{M} \to M$ called the *core map* such that

- $\text{cr}(\sigma) \geq \alpha$ and $\sigma(p_{\bar{M}}) = p_M$
- $\bar{M} = \tilde{h}^{n+1}_{M}(\alpha \cup \{p_{\bar{M}}\})$ where n is such that $\rho^{n+1}_{M} \leq \alpha < \rho^{n}_{M}$

If $\alpha = \rho^{n}_{M}$ then we call $\bar{M}$ the *n-th core of* M and write $\bar{M} = \text{core}_{n}(M)$. Similarly, if $\alpha = \rho^{\omega}_{M}$ then we call $\bar{M}$ the *core* and write $\bar{M} = \text{core}(M)$.

Fact 1.75. *Let* $\bar{M} = \text{core}_{\alpha}(M)$.

1. $\bar{M}$ *is sound above* α
2. $\rho^{n+1}_{\bar{M}} = \rho^{n+1}_{M}$ *where* n *is such that* $\rho^{n+1}_{M} \leq \alpha < \rho^{n}_{M}$
3. $\mathcal{P}(\rho^{n+1}_{M}) \cap \bar{M} = \mathcal{P}(\rho^{n+1}_{M}) \cap M$
4. *If* M *is iterable, then in coiteration of* M *against* $(M, \bar{M}, \alpha)$*, neither side involves a truncation,* $\bar{M}$ *is the root model on the phalanx side, and the coiteration ends in a common model.*

We now prove a lemma that will be useful in the proof of the condensation lemma, and extends the weak ISC in the case that the initial segment is just a measure.

Lemma 1.76. *Suppose* M *is a weakly iterable premouse. Let* $\alpha < \text{ht}(M)$ *be such that* $E^{M}_{\alpha} \neq \emptyset$ *and set* $\kappa = \text{cr}(E_{\alpha}), \tau = \kappa^{+M||\alpha}$. *Let* G *be the trivial completion of the normal measure determined by* E_{α} *and* κ. *Then,* $G = E^{M}_{\beta}$ *where* $\beta = \text{lh}(G) \leq \alpha$.

Proof. Using a reflection argument, we can assume that M is countable. By the Neeman-Steel lemma, fix an enumeration e of M and an e-minimal iteration strategy Σ for M. Let

$\bar{M} = (J^E_\beta, G)$ where $E = E^M$. We define an embedding from $\bar{M}$ to M which will allow us to coiterate the phalanx $(M, \bar{M}, \tau)$ with M, but note that it is possible that $\bar{M}$ and M are premice of different types, so we can not directly quote the previous lemmas.

Recall that any embeddings between premice are defined on the expansions of the premice, so let $\widehat{\bar{M}} = (\mathrm{Ult}(J^{\hat{E}}_\tau, G), \hat{G})$ and $\widehat{M} = (\mathrm{Ult}(J^{\hat{E}}_\tau, E_\alpha), \hat{E}_\alpha)$ be the expansions of $\bar{M}$ and M respectively, and define $\sigma : \widehat{\bar{M}} \to \widehat{M}$ by

$$\sigma(\pi_G(f)(\kappa)) = \pi_{E_\alpha}(f)(\kappa)$$

Note that by Łoś's theorem, this map is Σ_0-preserving and $\mathrm{cr}(\sigma) > \tau$. Also, $\rho^1_{\bar{M}} \leq \tau$ (it is the ultrapower of by a single measure of a sound structure) so this map σ is preserving enough to copy an iteration tree on the phalanx $(M, \bar{M}, \tau)$ to an iteration tree on M. Hence, $(M, \bar{M}, \tau)$ is iterable.

Let $(\bar{\mathcal{T}}, \mathcal{T})$ be the coiteration of $(M, \bar{M}, \tau)$ with M where we use Σ on the M-side and $\sigma\Sigma$ on the phalanx side. Moreover, let $\mathcal{T}'$ on M be the copied iteration $\sigma\bar{\mathcal{T}}$. We remark that this coiteration is unique in that if $\bar{M}$ is not an initial segment of M, then the first disagreement occurs at $\beta = \mathrm{ht}(\bar{M})$ and on the phalanx side we apply G. Because $\mathrm{cr}(G) = \kappa < \tau$, G is applied to M. Moreover, no extender $E^{\bar{\mathcal{T}}}_i$ is ever applied to $\bar{M}$. This is because $E^{\bar{\mathcal{T}}}_0 = G$, so $\nu^{\bar{\mathcal{T}}}_0 = \nu(G) = \tau$. Thus, if $\kappa^{\bar{\mathcal{T}}}_i < \nu^{\bar{\mathcal{T}}}_0$ then $\kappa^{\bar{\mathcal{T}}}_i < \tau$, so $E^{\bar{\mathcal{T}}}_i$ will be applied to M. Furthermore, no anomalies occur in the phalanx iteration as τ is a cardinal in M, which simplifies the argument to come.

Let $\theta + 1$ be the length of the coiteration. And for brevity, the models and iteration maps of $\mathcal{T}$ will be denoted as $M_i, \pi_{i,j}$ the models and iteration maps of $\bar{\mathcal{T}}$ will be denoted as $\bar{M}_i, \bar{\pi}_{i,j}$ and the models and iteration maps of $\mathcal{T}'$ will be denoted as $M'_i, \pi'_{i,j}$. Next, we show that

(a) $\bar{M}_\theta \trianglelefteq M_\theta$

(b) there is no truncation on the main branch of $\bar{\mathcal{T}}$

(c) the root model of the main branch of $\bar{\mathcal{T}}$ is $\bar{M}$

For (a), suppose this was not the case. Then, $M_\theta \lhd \bar{M}_\theta$, and there can be no truncation on the main branch of $\mathcal{T}$. Thus, the iteration map $\pi_{0,\Theta} : M \to M_\theta$ is total and Σ^*-preserving. However, letting $\sigma_\theta : \bar{M}_\theta \to M'_\theta$ be the copy map, we then have $\sigma_\theta \circ \pi_{0,\Theta} : M \to M'_\theta$ is a Σ^*-preserving map and $\sigma_\theta(M_\theta) \lhd M'_\theta$ is a non-simple Σ iterate of M which contradicts the Neeman-Steel lemma.

For (b), assume this was not the case. So, there is a truncation on the main branch of $\bar{\mathcal{T}}$. We already know that $\bar{M}_\theta \unlhd M_\theta$ and because there is a truncation $\bar{M}_\theta$ cannot be a proper initial segment of M_θ. So, $\bar{M}_\theta = M_\theta$ and there can be no truncation on the main branch of $\mathcal{T}$, so $\pi_{0,\theta}$ is total and Σ^*-preserving. Note, that the truncation on the main branch of $\bar{\mathcal{T}}$ will be copied to a truncation on the main branch of $\mathcal{T}'$, so M'_θ is a non-simple iterate of M. But then, again letting $\sigma_\theta : \bar{M}_\theta \to M'_\theta$ be the copy map, we have a Σ^*-preserving map $\sigma_\theta \circ \pi_{0,\theta} : M \to M'_\theta$ again contradicting the Neeman-Steel lemma.

Finally, for (c), assume that M is the root model on the main branch of $\bar{\mathcal{T}}$. We know there are no truncations along the main branch of $\bar{\mathcal{T}}$, so $\bar{\pi}_{-1,\theta} : M \to \bar{M}_\theta$ is total and Σ^*-preserving. It follows that $\bar{M}_\theta$ cannot be a proper initial segment of M_θ as then it would be a non-simple Σ-iterate of M. Similarly, there can be no truncation on $\mathcal{T}$ as then again, $\bar{M}_\theta = M_\theta$ would be a non-simple Σ-iterate of M. So, there are no truncations on the main branch of $\mathcal{T}$ and $M_\theta = \bar{M}_\theta$. By e-minimality and because $\pi_{-1,\theta} : M \to M_\theta$ is Σ^*-preserving,

$$\pi_{0,\theta}(e) \leq_{\text{lex}} \bar{\pi}_{-1,\theta}(e) \tag{1.9}$$

Again, let $\sigma_\theta : M_\theta \to M'_\theta$ be the copy map and so $\sigma_\theta \circ \bar{\pi}_{-1,\theta} : M \to M_\theta$ is Σ^*-preserving. By

e-minimality,

$$\sigma_\theta \circ \bar{\pi}_{-1,\theta}(e) = \pi'_{0,\theta}(e) \leq_{\text{lex}} \sigma_\theta \circ \pi_{0,\theta}(e)$$

Removing σ_θ from each side of the equality gives

$$\bar{\pi}_{-1,\theta}(e) \leq_{\text{lex}} \pi_{0,\theta}(e) \tag{1.10}$$

Hence, combining Equations 1.9 and 1.10 gives $\pi_{0,\theta} = \bar{\pi}_{-1,\theta}$. From this, in the usual way we can show that the first extenders used on either side of the coiteration are compatible, a contradiction.

Now, we know that $\bar{M}$ is the root model of the main branch of $\bar{\mathcal{T}}$, $\bar{M}_\theta \trianglelefteq M_\theta$, and there are no truncations along the main branch of $\bar{\mathcal{T}}$. Note, that by our remark that no extenders used in $\bar{\mathcal{T}}$ are applied to $\bar{M}$, this implies that $\bar{M} = \bar{M}_\theta$. We now consider the two cases of $\bar{M} \triangleleft \bar{M}_\theta$ or $\bar{M} = M_\theta$.

First, assume that $\bar{M} \triangleleft M_\theta$. We claim that M does not move, i.e. $M = M_\theta$. Suppose this is not the case. If E is the first extender used on the main branch of $\mathcal{T}$, then $\text{lh}(E) > \tau$ is a cardinal in M_θ. But note that $\rho^1_{\bar{M}} \leq \tau$ as $\bar{M}$ is a type I premouse. Hence, $\text{ht}(\bar{M}) < \text{lh}(E)$ so actually E is not applied. Thus, $M = M_\theta$ and $\bar{M} \triangleleft M$.

Now, assume that $\bar{M} = M_\theta$. We first show that there is no truncation on the main branch of $\mathcal{T}$. Suppose for contradiction that there was a truncation and let $M_{\gamma+1}$ be the result of the last truncation. Then, $\bar{M} = M_\theta$ is not sound as $\kappa_\gamma \notin \tilde{h}^\omega_{\bar{M}}(\rho^\omega_{\bar{M}} \cup \{p_{\bar{M}}\})$. However, $\bar{M}$ is sound above τ, so $\kappa_\gamma < \tau$. It follows that $M^* = M^*_{\gamma+1} \triangleleft M$. But then, $\rho^\omega_{M^*} \leq \kappa_\gamma < \tau$, contradicting that τ is a cardinal in M.

So, there is no truncation on the main branch of $\mathcal{T}$ and it follows that M is a type I premouse because $M_\theta = \bar{M}$ is a type I premouse. This implies $E_\alpha = G$, so $M = \bar{M}$. This completes the proof. We have shown that either $\bar{M} = M$ or $\bar{M} \lhd M$, and it follows that $G = E^M_\beta$. $\square$

As an application of the theory we have built thus far, we prove the Condensation lemma for premice with Mitchell-Steel indexing. Jensen orginally proved this for premice with λ-indexing. The fact that there are no superstrong extenders indexed on our premice simplifies the argument.

Theorem 1.77 (Condensation Lemma)**.** *Let M and $\bar{M}$ be premice of the same type and let $\sigma : \bar{M} \to M$ be a premice embedding which is both cardinal preserving and $\Sigma_0^{(n)}$-preserving for n such that $\sigma \restriction \rho^{n+1}_{\bar{M}} = \mathrm{id}$. Assume that M is weakly iterable and $\bar{M}$ is sound above $\nu = \mathrm{cr}(\sigma)$. Then, exactly one of the following holds*

(a) $\bar{M} = \mathrm{core}_\nu(M)$ and σ is the ν-th core map

(b) $\bar{M}$ is a proper initial segment of M

(c) $\bar{M} = \mathrm{Ult}^*(M||\delta, E^M_\alpha)$ where letting $\kappa = \mathrm{cr}(E^M_\alpha)$, $M||\delta$ is the longest initial segment of M such that $\nu = \kappa^{+M||\delta}$ and κ is the unique generator of E^M_α

(d) $\bar{M}$ is a proper initial segment of $\mathrm{Ult}^0(M, E^M_\nu)$

Remark 1.78. In Case (c), E^M_α having κ as its sole generator, could be restated as E^M_α is the trivial completion of the normal measure derived from E^M_α and κ.

Proof. Let us first note that the weak iterability of M guarantees that $\bar{M}$ is also weakly iterable, so it is solid by the solidity theorem. Moreover, by a reflection argument, we can assume the M is countable. Hence, we fix an e-minimal $(\omega, \omega_1 + 1)$-iteration strategy Σ for some enumeration e of M. Let $(\mathcal{T}, \bar{\mathcal{T}})$ be the coiteration of M against $(M, \bar{M}, \nu)$ where we use Σ on the M-side and the copied strategy $\sigma\Sigma$ on the phalanx side.

Notice that σ induces a phalanx embedding from $(M, \bar{M}, \nu)$ into M, and so by Lemma 1.71 the following are true:

- $\bar{M}$ is the root model on the main branch of $\bar{\mathcal{T}}$
- there are no truncations along the main branch of $\bar{\mathcal{T}}$
- $M_\theta^{\bar{\mathcal{T}}} \trianglelefteq M_\theta^{\mathcal{T}}$ where $\theta + 1 = \mathrm{lh}(\mathcal{T}) = \mathrm{lh}(\bar{\mathcal{T}})$

Let

- $M_k = M_k^{\mathcal{T}}$,$\bar{M}_k = M_k^{\bar{\mathcal{T}}}$, $E_k = E_k^{\mathcal{T}}$, $\bar{E}_k = E_k^{\bar{\mathcal{T}}}$, $\pi_{j,k} = \pi_{j,k}^{\mathcal{T}}$, $\bar{\pi}_{j,k} = \pi_{j,k}^{\bar{\mathcal{T}}}$
- $\kappa_k = \kappa_k^{\mathcal{T}}, \bar{\kappa}_k = \kappa_k^{\bar{\mathcal{T}}}$, $\tau_k = \tau_k^{\mathcal{T}}, \bar{\tau}_k = \tau_k^{\bar{\mathcal{T}}}, \alpha_k = \alpha_k^{\mathcal{T}}, \bar{\alpha}_k = \alpha_k^{\bar{\mathcal{T}}}$
- $\xi_k = \xi_k^{\mathcal{T}}, \bar{\xi}_k = \xi_k^{\bar{\mathcal{T}}}$, $\nu_k = \nu_k^{\mathcal{T}}, \bar{\nu}_k = \nu_k^{\bar{\mathcal{T}}}$

We consider three different cases for the proof. In the following, $\zeta + 1$ will be the result of the first ultrapower in $\mathcal{T}$. In other words, ζ is least with the property that $E_\zeta \neq \emptyset$. So, E_ζ comes from the M-sequence, and α_ζ is a cardinal in M_k for any $k > \zeta$. Additionally, if there is a truncation along the main branch of $\mathcal{T}$, then $\gamma + 1 \in [0, \theta]_T$ will denote the result of the last truncation.

Case 1: $M_\theta = \bar{M}_\theta$ and there is no truncation on the main branch of $\mathcal{T}$.
Let $\eta + 1$ be the result of the first ultrapower along the main branch of $\mathcal{T}$. Note that $\kappa_\eta \geq \rho_M^{n+1}$, as otherwise we would have

$$\kappa_\eta < \rho_M^{n+1} \implies \pi_{0,\theta}(\kappa_\eta) < \rho_{M_\theta}^{n+1} = \rho_{\bar{M}_\theta}^{n+1} = \rho_{\bar{M}}^{n+1}$$

Where the last equality holds because the iteration from $\bar{M}$ to $\bar{M}_\theta = M_\theta$ is above $\nu \geq \rho_{\bar{M}}^{n+1}$. But then, $\alpha_\eta < \pi_{0,\theta}(\kappa_\eta) < \rho_{\bar{M}}^{n+1} \leq \nu$ which is impossible.

Hence, the iteration from M to M_θ is above ρ_M^{n+1} and thus, all the models $\bar{M}, M$, and M_θ share a common value for the $(n+1)$-st projectum. Denote this common projectum by ρ^{n+1}.

Moreover, by solidity the standard parameters are preserved, namely

$$\pi_{0,\theta}(p_M) = p_{M_\theta} = \bar{\pi}_{0,\theta}(p_{\bar{M}}) \tag{1.11}$$

First, we show that σ is in fact $\Sigma_0^{(n)}$-cofinal. Suppose for a contradiction that σ is not. Then $\rho' = \sup \sigma[\rho^n_{\bar{M}}] < \rho^n_M$. Let A be $\Sigma_1^{(n)}(M)$ in p_M such that $a = A(-, p_M) \cap \rho^{n+1} \notin M$. Let A^* be $\Sigma_1^{(n)}(M_\theta)$ in p_{M_θ} and $\bar{A}$ be $\Sigma_1^{(n)} \bar{M}$ in $p_{\bar{M}}$ by the same definition as A. Equation 1.11 and the fact that both iterations are above ρ^{n+1} gives us

$$A(\xi, p_M) \longleftrightarrow A^*(\xi, p_{M_\theta}) \longleftrightarrow \bar{A}(\xi, p_{\bar{M}}) \qquad \text{for } \xi < \rho^{n+1} \tag{1.12}$$

Hence, $a = \bar{A}(-, p_{\bar{M}}) \cap \rho^{n+1}$. Let $\bar{B}$ be $\Sigma_0^{(n)}(\bar{M})$ such that

$$\bar{A}(u, v) \equiv (\exists z^n) \bar{B}(z^n, u, v)$$

Let $E = E^M$ and B be $\Sigma_0^{(n)}(M)$ by the same definition as $\bar{B}$. For $\xi < \rho^{n+1}$ we then have

$$\begin{aligned} \xi \in a \longleftrightarrow \bar{A}(\xi, p_{\bar{M}}) &\longleftrightarrow (\exists z^n) \bar{B}(z^n, \xi, p_{\bar{M}}) \\ &\longleftrightarrow (\exists z^n \in J^E_{\rho'}) B(z^n, \xi, \sigma(p_{\bar{M}})) \end{aligned}$$

Hence, $a \in M$ as it is a $\boldsymbol{\Sigma}_0^{(n)}(M)$ subset of ρ^{n+1}. This contradicts the choice of A, so σ is $\Sigma_0^{(n)}$-cofinal (in particular, it is $\Sigma_1^{(n)}$-preserving).

Next we show that $\sigma(p^{n+1}_{\bar{M}}) = p^{n+1}_M$. By solidity and Lemma 1.8, this amounts to showing that $\sigma(p^{n+1}_{\bar{M}}) \in P^{n+1}_M$. Assume this were not the case, and note that because $\bar{M}$ is solid, every $\alpha \in \sigma(p^{n+1}_{\bar{M}})$ has a generalized witness with respect to M as σ is $\Sigma_1^{(n)}$-preserving. Hence, $\sigma(p^{n+1}_{\bar{M}}) <^* p^{n+1}_M$ where $<^*$ is the canonical ordering on finite sets of decreasing ordinals. Let

α be such that $\alpha \in p_M^{n+1} - \sigma(p_{\bar{M}}^{n+1})$ and $\sigma(p_{\bar{M}}^{n+1}) - (\alpha+1) = p_M^{n+1} - (\alpha+1)$. Then,

$$\sigma(p_{\bar{M}}) - (\alpha+1) = \sigma(p_{\bar{M}}^{n+1}) - (\alpha+1) = p_M^{n+1} - (\alpha+1) = p_M - (\alpha+1) \tag{1.13}$$

In particular,

$$\sigma(p_{\bar{M}}) \subseteq \alpha \cup p_M - (\alpha+1) \tag{1.14}$$

Let W_α be the $\Sigma_1^{(n)}$-hull of $\alpha \cup p_M - (\alpha+1)$ and $\sigma_\alpha : W_\alpha \to M$ be the canonical witness map which is $\Sigma_1^{(n)}$-preserving. By Equation 1.14 it follows that $\sigma(p_{\bar{M}}) \in W_\alpha$. Thus, because $\bar{M}$ is sound above ν and σ_α is $\Sigma_1^{(n)}$-preserving we have

$$\text{rng}(\sigma) = \tilde{h}_M^{n+1}(\nu \cup \{\sigma(p_{\bar{M}})\}) \subseteq \text{rng}(\sigma_\alpha) \tag{1.15}$$

Let $X = \sigma_\alpha^{-1}[\text{rng}(\sigma)]$ and note that

$$X = \tilde{h}_{W_\alpha}^{n+1}(\nu \cup \{\bar{p}\})$$

where $\bar{p} = \sigma_\alpha^{-1}(\sigma(p_{\bar{M}}))$ Because $X \cong \text{rng}(\sigma)$, by collapsing X we obtain a $\Sigma_1^{(n)}$-preserving map $\bar{\sigma} : \bar{M} \to W_\alpha$ and $\bar{\sigma}$ satisfies $\sigma = \sigma_\alpha \circ \bar{\sigma}$.

Let A be $\Sigma_1^{(n)}(M)$ in p_M such that $a = A(-, p_M) \cap \rho^{n+1} \notin M$. Let $\bar{A}$ be $\Sigma_1^{(n)}(\bar{M})$ in $p_{\bar{M}}$ by the same definition. As in Equation 1.12, we obtain $\bar{A}(-, p_{\bar{M}}) \cap \rho^{n+1} = a$. Now, let A^* be $\Sigma_1^{(n)}(W_\alpha)$ in $\bar{p} = \bar{\sigma}(p_{\bar{M}})$ by the same definition as A and $\bar{A}$. Note that $\text{cr}(\bar{\sigma}) = \nu \geq \rho^{n+1}$, so for $\xi < \rho^{n+1}$, we have

$$\xi \in a \longleftrightarrow \bar{A}(\xi, p_{\bar{M}}) \longleftrightarrow A^*(\xi, \bar{p})$$

So, a is a $\Sigma_1^{(n)}(W_\alpha)$ subset of ρ^{n+1}. But $W_\alpha \in M$ and so $a \in M$. Contradiction.

Wrapping this first case up, recall that we have shown that $\sigma : \bar{M} \to M$ is $\Sigma_1^{(n)}$-preserving and $\sigma(p_{\bar{M}}^{n+1}) = p_M^{n+1}$. The latter implies that $\mathrm{rng}(\sigma) \cap P_M^* \neq \emptyset$ as $\mathrm{cr}(\sigma) \geq \rho^{n+1}$. Hence, by Downward Extensions of Embeddings (cf. Lemma 1.11) σ is Σ^*-preserving and $\sigma(p_{\bar{M}}) = p_M$. Thus, σ satisfies the conditions of a core map. $\square$ Case 1

From now on, we can assume that either $\bar{M}_\theta \lhd M_\theta$ or there is a truncation on the main branch of $\mathcal{T}$.

Case 2: $\bar{M}$ is fully sound

In this case, we show that either option (b) or option (d) of the Condensation lemma holds. Recall the notation that $\zeta + 1$ is the result of the first ultrapower on $\mathcal{T}$. By the assumption mentioned prior the start of this case, either $\bar{M}_\theta \lhd M_\theta$ or there is a truncation on the main branch of $\mathcal{T}$.

Subcase (i) Assume that $\alpha_\zeta > \nu$. We show $\bar{M} \lhd M$.

Assume first that $\bar{M}_\theta \lhd M_\theta$. This implies that $\bar{M} = \bar{M}_\theta$. This is because $\bar{M}$ projects to ν and any extender from $\bar{\mathcal{T}}$ applied to $\bar{M}$ would have critical point at least ν so the resulting ultrapower would not be sound. In addition, α_ζ is a cardinal in M and $\alpha_\zeta > \nu$. So, it must be the case that $\bar{M} \unlhd M_\theta | \alpha_\zeta$ as $\bar{M}$ projects to ν. By the agreement in an iteration, we have $M|\alpha_\zeta = M_\theta|\alpha_\zeta$. It follows that $\bar{M} \unlhd M|\alpha_\zeta$ so $\bar{M} \lhd M$.

Now assume that there is a truncation on the main branch of $\mathcal{T}$. Moreover, we can assume that $M_\theta = \bar{M}_\theta$ as we have taken care of the case $\bar{M}_\theta \lhd M_\theta$ already. Recall $\gamma + 1 \in [0, \theta]_T$ is the result of the last truncation along the main branch of T. Let $M^* = M_{\gamma+1}^*$ be the truncated model. We claim that $M^* = \bar{M}$. To see this, notice that due to the truncation $\rho_{M^*}^\omega \leq \kappa_\gamma$. It follows that $\rho_{M^*}^\omega = \rho_{M_\theta}^\omega$ and $p_{M_\theta} = \pi_{\xi_j,\theta}(p_{M^*})$ by solidity and because the iteration from M^*-to-M_θ is above κ_γ. On the $\bar{M}$-side we have a similar result: $\bar{M}$ is solid, projects to ν and the iteration from $\bar{M}$-to-M_θ is above ν, so $\rho_{\bar{M}}^\omega = \rho_{M_\theta}^\omega$ and $p_{M_\theta} = \bar{\pi}_{0,\theta}(p_{\bar{M}})$. Hence, $\bar{M}$, M^* and M_θ share the same ultimate projectum. Denote this common projectum by ρ. Because both $\bar{M}$ and M^* are fully sound, letting n such that ρ is the $(n+1)$-st projectum we have

$$\operatorname{rng}(\bar{\pi}_{0,\theta}) = \tilde{h}^{n+1}_{M_\theta}(\rho \cup \{p_{M_\theta}\}) = \operatorname{rng}(\pi_{\xi_\gamma,\theta})$$

Hence, by collapsing, $\bar{M} = M^*$.

We now proceed as above. If $M_{\xi_\gamma} = M$ then we are done. Otherwise, α_ζ is a cardinal in M_{ξ_γ}. Because $M^* = \bar{M}$ projects to ν it follows that $\operatorname{ht}(\bar{M}) \leq \alpha_\zeta$. Hence, by the agreement between models in the iteration, we get $\bar{M} \trianglelefteq M_{\xi_\gamma}|\alpha_\zeta = M|\alpha_\zeta$, so again $\bar{M} \triangleleft M$.

Subcase (ii) Now assume that $\alpha_\zeta = \nu$. We show $\bar{M} \triangleleft \operatorname{Ult}^0(M, E^M_\nu)$.

Note this implies that $\zeta = 0$ and $\alpha_0 = \nu$. Then, $M_1 = \operatorname{Ult}^*(M, E^M_\nu)$ and notice the arguments from the previous subcase show that $\bar{M} \triangleleft M_1$. As $\bar{M}$ projects to ν, we have $\operatorname{ht}(\bar{M}) < \nu^{+M_1}$ (we also have $\rho^\omega_{\bar{M}} = \nu$ but this fact is not needed). Now, if $\pi : M \to \operatorname{Ult}^0(M, E^M_\nu)$ denotes the 0-ultrapower map, then note that $\pi(\kappa_0) = \pi_{0,1}(\kappa_0)$ and $\operatorname{Ult}^0(M, E^M_\nu)$ and $M_1 = \operatorname{Ult}^*(M, E^M_\nu)$ agree up to this common image. Moreover, $\pi(\kappa_0) > \alpha_0 = \nu$ and in particular $\pi(\kappa_0) \geq \nu^{+M_1}$. Hence, $\bar{M} \triangleleft M_1|\pi(\kappa_0) = \operatorname{Ult}^0(M, E^M_\nu)|\pi(\kappa_0)$.

□ Case 2

Case 3: $\bar{M}$ is not fully sound

Recall our assumption that either $\bar{M}_\theta \triangleleft M_\theta$ or there is a truncation on the main branch of $\mathcal{T}$. Note that $\bar{M}$ not being sound implies that $\bar{M}_\theta$ is not sound as there are no truncations along $[0, \theta]_{\bar{T}}$. Hence, we cannot have $\bar{M}_\theta \triangleleft M_\theta$, so in this case there is a truncation along the main branch of $\mathcal{T}$ and $M_\theta = \bar{M}_\theta$.

We show that conclusion (c) holds. Recall $\gamma + 1$ is the result of the last truncation on the main branch of $\mathcal{T}$. Let $M^* = M^{*\mathcal{T}}_{\gamma+1}$ and $\xi^* = \xi_\gamma$. Notice that $\kappa_\gamma \geq \rho^{n+1}_{M^*}$. If this were not the case then $\pi_{\xi^*,\theta}(\kappa_\gamma) < \rho^{n+1}_{M_\theta} = \rho^{n+1}_{\bar{M}_\theta} = \rho^{n+1}_{\bar{M}}$ and so $\alpha_\gamma < \pi_{\xi^*,\theta}(\kappa_\gamma) < \rho^{n+1}_{\bar{M}} \leq \nu$ which is impossible. Hence, the iteration from ξ^* to θ on $\mathcal{T}$ is above the $(n+1)$-st projectum so as in Case 1 we set $\rho^{n+1} := \rho^{n+1}_{M^*} = \rho^{n+1}_{M_\theta} = \rho^{n+1}_{\bar{M}}$.

First we show that $M^* \lhd M$. We claim that $\kappa_\gamma < \nu$. If not, then $\kappa_\gamma \geq \nu \geq \rho^{n+1}$ and we could conclude that $M^* = \bar{M}$. This is because both M^* and $\bar{M}$ are sound above ν and both iterations are above ν and preserve the standard parameters. So, we would have

$$\mathrm{rng}(\pi_{\xi^*,\theta}) = \tilde{h}^{n+1}_{M_\theta}(\nu \cup \{p_{M_\theta}\}) = \mathrm{rng}(\bar{\pi}_{0,\theta})$$

Which implies that $M^* = \bar{M}$. However, M^* is fully sound and we are assuming $\bar{M}$ is not, so this rules out the possibility that $\kappa_\gamma \geq \nu$. Hence, $\alpha_\zeta \geq \nu > \kappa_\gamma$ where as above, $\zeta + 1$ is the result of the first ultrapower of $\mathcal{T}$. Let us first check that $\kappa_\gamma < \nu_\zeta$. If not, then $\kappa_\gamma = \nu_\zeta < \alpha_\zeta$. It follows that $\alpha_\zeta \leq \nu_{\zeta+1} < \alpha_{\zeta+1}$, so E_γ will be applied to $M_{\zeta+1}$. But then, α_ζ is a cardinal in $M_{\zeta+1}$ and $\tau_\gamma \leq \alpha_\zeta$. Hence, if E_γ is applied to $M_{\zeta+1}$ we would not truncate. So, E_γ will be applied to $M_\zeta = M$. Thus, $M^* \lhd M$.

Next, we show that κ_γ is the largest cardinal in $M|\nu$. Suppose not and say $\tau = \kappa^{+M|\nu}$. By acceptability, τ is a cardinal in $\bar{M}$ because ν is a cardinal in $\bar{M}$. Moreover, because σ is cardinal preserving, $\tau = \sigma(\tau)$ is also a cardinal in M. By the agreement in models in an iteration, the cardinal structure below ν remains the same in M_k for any $k \geq 0$. It follows that $\tau_\gamma = \tau$, but that means we would not truncate, a contradiction.

Let us now argue that $\alpha_\zeta > \nu$. If not, then $\zeta = 0$ and $\alpha_0 = \nu$. Moreover, $\nu = \tau_\gamma$ and $M^* = M||\nu$ as ν is collapsed in $M|(\nu + 1)$. Note that M^* is fully sound and $\rho^\omega_{M^*} = \kappa_\gamma$ as M^* cannot project across κ_γ because $\kappa_\gamma = \sigma(\kappa_\gamma)$ is a cardinal in M. So, $p_{M^*} \subseteq [\kappa_\gamma, \nu)$ and thus, $p_{M_\theta} \cap \pi_{0,\theta}(\kappa_\gamma) = \emptyset$. Note that $\pi_{0,\theta}(\kappa_\gamma) > \alpha_\gamma > \alpha_0 = \nu$. So, p_{M_θ} has no ordinals in the interval $[\kappa, \nu]$. Because $\rho^\omega_{M_\theta} = \kappa_\gamma$ and the iteration from $\bar{M}$ to M_θ is above ν, $\rho^\omega_{\bar{M}} = \kappa_\gamma$ as well. $\bar{M}$ is sound above ν, but not fully sound, so $p_{\bar{M}}$ will have ordinals in the interval

$[\kappa, \nu]$. Because $\mathrm{cr}(\bar{\pi}_{0,\theta}) > \nu$, $\bar{\pi}_{0,\theta}(p_{\bar{M}}) \cap (\nu+1) = p_{M_\theta} \cap (\nu+1)$. However, this contradicts that $p_{M_\theta} \cap [\kappa, \nu] = \emptyset$. Thus, $\alpha_\zeta > \nu$.

We claim that

$$\bar{M} = \mathrm{Ult}^*(M^*, U) \qquad \text{where } U \text{ is the normal measure on } \kappa_\gamma \text{ derived from } E_\gamma \tag{1.16}$$

To see this, let $\bar{M}' = \mathrm{Ult}^*(M^*, U)$ and $\pi^0 : M^* \to \bar{M}'$ the ultrapower map. We show that $\bar{M}' = \bar{M}$ which gives us Equation 1.16. Let $\pi^1 : \bar{M}' \to M_{\gamma+1}$ be the factor map defined by $\pi^1(\pi^0(f)(\kappa_\gamma)) = \pi_{0,\gamma+1}(f)(\kappa_\gamma)$ (this is a slight abuse of notation as literally E_γ is an extender on $\mathcal{P}([\kappa_\gamma]^{<\omega})$ but it should be obvious how to modify this to the correct definition).

This results in the following commutative diagram

$$\begin{tikzcd} \bar{M}' \arrow[dr, "\pi^1"] & \\ M^* \arrow[u, "\pi^0"] \arrow[r, "\pi_{0,\gamma+1}"] & M_{\gamma+1} \end{tikzcd}$$

Note that $\kappa_\gamma < \rho^n_{M^*}$. To see this, recall that $\nu < \rho^n_{\bar{M}}$ so that $\nu \leq \bar{\pi}_{0,\theta}(\nu) < \rho^n_{M_\theta}$. If $\rho^n_{M^*} \leq \kappa_\gamma$ then $\rho^n_{M_\theta} \leq \rho^n_{M^*} < \kappa_\gamma$. But then $\rho^n_{M_\theta} < \kappa_\gamma < \nu < \rho^n_{M_\theta}$, a contradiction. Hence, $\rho^{n+1} \leq \kappa_\gamma < \rho^n_{M^*}$. It follows that the ultrapower maps π^0 and $\pi_{0,\gamma+1}$ are $\Sigma_0^{(n)}$-cofinal because U, E_γ are weakly amenable. Hence, using Łoś's 's theorem, π^1 is $\Sigma_0^{(n)}$-preserving. However, π^1 is actually cofinal at the n-th level as well, as for $\xi < \rho^n_{M^*}$ we have $\pi^1 \circ \pi^0(\xi) = \pi_{0,\gamma+1}(\xi)$. Hence, $\sup \pi^1[\rho^n_{\bar{M}'}] = \sup \pi_{0,\gamma+1}[\rho^n_{M^*}] = \rho^n_{M_{\gamma+1}}$. So, π^1 is $\Sigma_0^{(n)}$-cofinal.

Note that $\bar{M}' = \tilde{h}^{n+1}_{\bar{M}'}(\rho^{n+1} \cup \{\kappa_\gamma, \pi^0(p_{M^*})\})$ as M^* is sound and $\bar{M}'$ is the ultrapower by a single measure. Hence, $\mathrm{rng}(\pi^1) = \tilde{h}^{n+1}_{M_{\gamma+1}}(\rho^{n+1} \cup \{\kappa_\gamma, \pi_1 \circ \pi_0(p_{M^*})\})$ because $\mathrm{cr}(\pi^1) \geq \nu > \kappa_\gamma$. Note that by the weak amenability of U, $\nu = \kappa_\gamma^{+M^*} = \kappa_\gamma^{+\bar{M}'}$ so that ν is contained in this hull. Moreover, by the commutativity of the maps, $\pi^1 \circ \pi^0(p_{M^*}) = \pi_{0,\gamma+1}(p_{M^*}) = p_{M_{\gamma+1}}$ Thus, $\mathrm{rng}(\pi^1) = \tilde{h}^{n+1}_{M_{\gamma+1}}(\nu \cup \{p_{M_{\gamma+1}}\}))$. It follows that $\mathrm{rng}(\pi_{\gamma+1,\theta} \circ \pi^1) = \tilde{h}^{n+1}_{M_\theta}(\nu \cup \{p_{M_\theta}\}))$ as

$\operatorname{cr}(\pi_{\gamma+1,\theta}) \geq \nu$. Finishing things off, recall that $\bar{M}$ is solid and sound above ν so $\operatorname{rng}(\bar{\pi}_{0,\theta}) = \tilde{h}^{n+1}_{M_\theta}(\nu \cup \{p_{M_\theta}\})$. Thus, $\operatorname{rng}(\bar{\pi}_{0,\theta}) = \tilde{h}^{n+1}_{M_\theta}(\nu \cup \{p_{M_\theta}\})) = \operatorname{rng}(\pi_{\gamma+1,\theta} \circ \pi^1)$, and collapsing gives $\bar{M} = \bar{M}'$.

Next, we show that we can replace U with some extender E^M_β on the M-sequence. First, we show that either E_γ is not the top extender of M_γ or else there is a truncation $\beta+1 \in [0,\gamma]_T$. Suppose towards a contradiction that this is not the case. So, $E_\gamma = E^{M_\gamma}_{\text{top}}$ and there are no truncations on $[0,\gamma]_T$. Hence, $\pi_{0,\gamma}$ is a total map. Let $\gamma^* + 1$ be the result of the first ultrapower on $[0,\gamma]_T$ so that $M_{\gamma^*+1} = \operatorname{Ult}^*(M, E_{\gamma^*})$. Note that we cannot have $\kappa_\gamma \leq \kappa_{\gamma^*}$ as this would imply that $\mathcal{P}(\kappa_\gamma) \cap M = \mathcal{P}(\kappa_\gamma) \cap M_\gamma$. But if this were the case, we would not truncate when applying E_γ.

So, $\kappa_\gamma > \kappa_{\gamma^*}$ and since $\kappa_\gamma = \operatorname{cr}(E^{M_\gamma}_{\text{top}})$, there is some $\bar{\kappa} \in M$ such that $\kappa_\gamma = \pi_{0,\gamma}(\bar{\kappa})$. Clearly, $\bar{\kappa} \geq \kappa_{\gamma^*}$ because $\operatorname{cr}(\pi_{0,\gamma}) = \kappa_{\gamma^*}$. Because we apply E_γ to $M^* \lhd M$, this gives

$$\alpha_{\gamma^*} < \pi_{0,\gamma}(\kappa_{\gamma^*}) \leq \pi_{0,\gamma}(\bar{\kappa}) = \kappa_\gamma < \nu_\zeta$$

where ζ still denotes the least i such that $E_i \neq \emptyset$. This gives $\alpha_{\gamma^*} < \alpha_\zeta$, contradicting normality of the iteration.

Thus, either $E_\gamma \in M_\gamma$ or there is a truncation $\beta + 1 \in [0,\gamma]_T$. We first tackle the situation where $E_\gamma \in M_\gamma$. Note, that by Lemma 1.76, there is a $\alpha < \alpha_\gamma = \operatorname{lh}(E_\gamma)$ such that $E = E^{M_\gamma}_\alpha$ is the trivial completion of the normal measure generated by E_γ and κ_γ. Hence, E is the trivial completion completion of U. If $M_\gamma = M$, then of course E is on the M-sequence. If not, then $\alpha_\zeta > \nu$ is a cardinal in $M_\gamma || \alpha_\gamma$. Moreover, $\nu = \kappa_\gamma^{+M_\gamma || \alpha_\gamma}$, so actually

$$\alpha_\zeta \geq \nu^{+M_\gamma || \alpha_\gamma} \tag{1.17}$$

Notice, $\alpha < \nu^{+M_\gamma || \alpha_\gamma}$. This is because $\alpha = \operatorname{lh}(E)$ and E is the trivial completion of the measure U, so $\nu(E) = \nu = \kappa_\gamma^{+M||\gamma}$. Hence, $\text{card}^{M_\gamma||\alpha_\gamma}(\alpha) = \nu$, so $\alpha < \nu^{+M_\gamma||\alpha_\gamma}$. By Equation 1.17, $\alpha < \alpha_\zeta$ and so E is on the M-sequence as $M|\alpha_\zeta = M_\gamma|\alpha_\zeta$.

Now, assume there is a truncation on the branch $[0,\gamma]_T$ and let $\beta+1$ denote the result of the last truncation. Let $M' = M^{*\mathcal{T}}_{\beta+1}$. We claim that the normal measure derived from $E^{M'}_{\text{top}}$ and κ_γ is U. In this case, $E_\gamma = E^{M_\gamma}_{\text{top}}$ so letting $\xi = \xi_\beta$, $\kappa_\gamma = \pi_{\xi,\gamma}(\kappa')$ where $\kappa' = \text{cr}(E^{M_\xi}_{\text{top}})$. Note that we cannot have $\kappa_\beta \leq \kappa'$. If this was the case, then

$$\alpha_\beta < \pi_{\xi,\gamma}(\kappa_\beta) \leq \pi_{\xi,\gamma}(\kappa') = \kappa_\gamma < \alpha_\zeta$$

which contradicts normality of the iteration. Hence, $\kappa' < \kappa_\beta$ so that $\kappa_\gamma = \kappa' = \text{cr}(E^{M'}_{\text{top}})$. As κ_β is a limit cardinal in M' and all later models we get that $\kappa_\gamma^{+M'} = \kappa_\gamma^{+M_\gamma} = \nu$. Hence, for $x \in \mathcal{P}(\kappa_\gamma) \cap M' = \mathcal{P}(\kappa_\gamma) \cap M_\gamma$ we have $\pi_{\xi,\gamma}(x) = x$ so that

$$\kappa_\gamma \in E^{M'}_{\text{top}}(x) \longleftrightarrow \pi_{\xi,\gamma}(\kappa_\gamma) \in E^{M_\gamma}_{\text{top}}(\pi_{\xi,\gamma}(x)) \longleftrightarrow \kappa_\gamma \in E^{M_\gamma}_{\text{top}}(x) \longleftrightarrow x \in U$$

Thus, U is the normal measure derived from $E^{M'}_{\text{top}}$ and $\text{cr}(E^{M'}_{\text{top}}) = \kappa_\gamma$.

We can now proceed in the same way as in the situation that $E_\gamma \in M_\gamma$. As above, let E denote the trivial completion of U and α be its length. In this case, noting that $M' \lhd M_\xi$, by Lemma 1.76 $E = E^{M_\xi}_\alpha$ and $\alpha \leq \text{ht}(M')$. If $M_\xi = M$ or if $\alpha_\zeta > \text{ht}(M')$, then clearly E is on the M-sequence. So assume $M_\xi \neq M$ and $\alpha_\zeta < \text{ht}(M')$. Then, α_ζ is a cardinal in M_ξ and M'. Note that $\nu = \kappa_\gamma^{+M'} < \alpha_\zeta$, so by acceptability $\nu = \kappa_\gamma^{+M_\xi}$ as well. We then have $\text{card}^{M_\xi}(\alpha) < \nu^{+M_\xi} \leq \alpha_\zeta$ as above. Because $M|\alpha_\zeta = M_\xi|\alpha_\zeta$ this gives E is on the M-sequence.

Hence, we have shown that there is an extender $E = E^M_\alpha$ such that E is the trivial completion of U. By Equation 1.16, it follows that $\bar{M} = \text{Ult}^*(M^*, E)$. To finish this case off, recall that $\text{cr}(E) = \kappa_\gamma$, $\nu = \kappa_\gamma^{+M^*}$ and by the rules of normal iteration, M^* was the longest initial segment $M||\delta$ of M such that ν is the cardinal successor of κ_γ in $M||\delta$. □ Case 3

We have considered all possible cases, and have thus completed the proof. □

1.6 Thick Classes and Universal Weasels

We review the important definitions and theorems related to thick classes and universal weasels. The constructions of Steel's K as outlined in *Core Model Iterability Problem* depends on the anti-large cardinal hypothesis:

The definitions and theorems outlined here are all taken from section 2 of Schimmerling's paper [11], but they are due to Steel (cf. [14]). Following Steel, the setting in which we work adds an additional technical hypothesis:

Technical Hypothesis Ω is a measurable cardinal and μ is a normal measure over Ω.

Definition 1.79. Let

$$A_1 = \{\kappa < \Omega \mid \kappa \text{ is an inaccessible cardinal and } \kappa^{+\mathsf{K}^c} = \kappa^+\}$$

and

$$A_0 = \{\lambda \in A_1 \mid A_1 \cap \lambda \text{ is not stationary in } \lambda\}$$

Definition 1.80. A *weasel* is an $\Omega + 1$ iterable premouse of height Ω.

Theorem 1.81 (Comparison Lemma)**.** *If $(\mathcal{S}, \mathcal{T})$ is the coiteration of two mice (P, Q) and the length of the length of the coiteration is θ. Then at least one of the following is true,*

1. *$M_\theta^{\mathcal{S}} \trianglelefteq M_\theta^{\mathcal{T}}$ and there are no truncations on the main branch of $\mathcal{S}$. Moreover, if $\theta = \Omega$ then P is a weasel and $\pi_{0,\Omega}^{\mathcal{S}}[\Omega] \subseteq \Omega$.*

2. *$M_\theta^{\mathcal{T}} \trianglelefteq M_\theta^{\mathcal{S}}$ and there are no truncations on the main branch of $\mathcal{T}$. Moreover, if $\theta = \Omega$ then Q is a weasel and $\pi_{0,\Omega}^{\mathcal{T}}[\Omega] \subseteq \Omega$.*

In the coiteration $(\mathcal{S}, \mathcal{T})$ of (P, Q), if $M_\theta^{\mathcal{T}} \trianglelefteq M_\theta^{\mathcal{S}}$ we say P *does not lose* in the coiteration and will use $P \leq^* Q$ as a notational shorthand to express this.

Definition 1.82. A weasel Q is universal iff $P \leq^* Q$ for all $\Omega + 1$ iterable premice of height $\leq \Omega$.

Definition 1.83. Let Q be a weasel and $\Gamma \subseteq Q$. Then, Γ is *thick in* Q iff there is a club C in Ω such that for all $\lambda \in A_0 \cap C$,

1. $\lambda^{+Q} = \lambda^+$
2. λ is not the critical point of a total-on-Q extender on the Q-sequence
3. there is a λ-club in $\Gamma \cap \lambda^+$

Definition 1.84. We say Q is a *thick weasel* iff Ω is thick in Q.

Theorem 1.85. *Suppose that $\pi : P \to Q$ is an elementary embedding and that* $\mathrm{rng}(pi)$ *is thick in Q. Let $\Delta = \{\alpha < \Omega \mid \pi(\alpha) = \alpha\}$. Then, Δ is thick in both P and Q.*

Theorem 1.86. *Let $\mathcal{T}$ be an iteration tree on a thick weasel Q with* $\mathrm{lh}(\mathcal{T}) = \theta + 1 \leq \Omega + 1$. *Assume that there are no truncations along $[0, \theta]_T$ and $\pi_{0,\theta}^{\mathcal{T}}[\Omega] \subseteq \Omega$. Let $\Delta = \{\alpha < \Omega \mid \pi(\alpha) = \alpha\}$. Then, Δ is thick in both Q and $M_\theta^{\mathcal{T}}$.*

Definition 1.87. A thick weasel Q has the *definability property* at α iff $\alpha \in \mathrm{Hull}^Q(\alpha \cup \Gamma)$ for any Γ which is thick in Q.

Definition 1.88. A thick weasel Q has the *hull property* at α iff

$$\mathcal{P}(\alpha) \cap Q \subseteq \text{ the Mostowski collapse of } \mathrm{Hull}^Q(\alpha \cup \Gamma)$$

for any Γ which is thick in Q.

Theorem 1.89. *Let $\beta < \Omega$ and assume Q is a thick weasel with the hull and definability property at all $\alpha < \beta$. Suppose that $\mathcal{T}$ is an iteration tree on Q with $\mathrm{lh}(T) = \theta + 1 \leq \Omega + 1$. Assume there are no truncations along $[0, \theta]_T$ and $\pi^{\mathcal{T}}_{0,\theta}[\Omega] \subseteq \Omega$. Then, the following hold for all $\alpha < \beta$*

1. *$M^{\mathcal{T}}_\theta$ does not have the definability property at α iff there exists an $\eta + 1 \in [0, \theta]_T$ such that α is a generator of $E^{\mathcal{T}}_\eta$.*

2. *$M^{\mathcal{T}}_\theta$ does not have the definability property at α iff there exists an $\eta + 1 \in [0, \theta]_T$ such that*

$$\mathrm{cr}(E^{\mathcal{T}}_\eta)^{+M^{\mathcal{T}}_\theta} \leq \alpha < \nu(E^{\mathcal{T}}_\eta)$$

Definition 1.90. Let P be a mouse of height $< \Omega$. Then, P is A_0-*sound* iff there exists a thick weasel Q such that $P \lhd Q$ and Q has the definability property at all $\alpha < \mathrm{ht}(P)$. We call Q a *soundness witness* for P.

Theorem 1.91. *Let P, Q be A_0-sound mice. Then $P \unlhd Q$ or $Q \unlhd P$.*

Definition 1.92. K is the union of all A_0-sound mice.

Chapter 2

Covering Theorems

In this section we prove Theorems 2.3 and 2.22.

Definition 2.1. Given a premouse M, and an extender E^M_α on the M-sequence, we will say the *order* of E^M_α is β and write $o^M_*(E^M_\alpha) = \beta$ iff

$$\mathrm{otp}\{\xi < \alpha \mid E^M_\xi \neq \emptyset \text{ and } \mathrm{cr}(E^M_\xi) = \mathrm{cr}(E^M_\alpha)\} = \beta$$

Remark 2.2. This definition of order does not precisely match up with the Mitchell order o, but if $o^M_*(E^M_\alpha) = \beta$ then $o^M(E^M_\alpha) \geq \beta$.

Theorem 2.3. *Assume $\kappa > \omega_2$ is a regular cardinal in K, but $\mathrm{cf}(\kappa) = \omega$. If $\mathrm{card}(\kappa)$ is ω-closed, then κ is measurable in K.*

Remark 2.4. By ω-closed, we mean $\lambda^\omega < \mathrm{card}(\kappa)$ for all $\lambda < \mathrm{card}(\kappa)$.

We prove theorem 2.3 through a sequence of lemmas. Let W be a soundness witness for $\mathsf{K}|\Lambda$ for $\Lambda >> \kappa$. First, we construct an elementary substructure X with all the relevant information. Fix $\kappa > \omega_2$ as in the statement of the theorem. Build an elementary substructure $X \prec (H_\Gamma, \in)$ where $\Gamma >> \Omega$ such that

- $\mathrm{card}(X) < \mathrm{card}(\kappa)$
- $\sup(X \cap \kappa) = \kappa$
- ${}^{\omega}X \subseteq X$
- $\Omega, \mu, W, etc \in X$

Let $\sigma_0 : H \to X$ be the inverse to the transitive collapse. Define $\bar{W} = \sigma_0^{-1}(W)$ and $\sigma : \bar{W} \to W = \sigma_0 \upharpoonright \bar{W}$. Let $(\bar{\mathcal{T}}, \mathcal{T})$ be the coiteration of $(\bar{W}|\bar{\kappa}, W)$.

Based on the arguments in the proof of the weak covering lemma (cf. [9]), we know

Fact 2.5. *Let $\Theta + 1$ be the length of coiteration $(\bar{\mathcal{T}}, \mathcal{T})$. Then, $\bar{W}|\bar{\kappa}$ does not move in the coiteration. Hence, $\bar{W}|\bar{\kappa} \lhd W_\Theta$, where W_Θ is the last model on $\mathcal{T}$.*

For brevity, we will set

- $W_k = M_k^{\mathcal{T}}$, $E_k = E_k^{\mathcal{T}}$, $\pi_{j,k} = \pi_{j,k}^{\mathcal{T}}$
- $\kappa_k = \kappa_k^{\mathcal{T}}$, $\tau_k = \tau_k^{\mathcal{T}}, \alpha_k = \alpha_k^{\mathcal{T}}$, $\xi_k = \xi_k^{\mathcal{T}}, \nu_k = \nu_k^{\mathcal{T}}$

Remark 2.6. For the next three lemmas we will argue assuming there is a truncation on the main branch of $\mathcal{T}$. If there is not a truncation on the main branch of $\mathcal{T}$, then the arguments to come can be adapted in a standard way: we replace arguments using the fine structure of a mouse with arguments using the hull and definability properties of a weasel.

An alternative to this approach would be to use λ-indexing. Using λ-indexing, it is known that in the above coiteration there will be a truncation on the main branch of $\mathcal{T}$. Whether or not there is a truncation on the main branch of $\mathcal{T}$ using Mitchell-Steel indexing is an open question.

Lemma 2.7. *The ordinal $\bar{\kappa}$ is not definably singularized over W_Θ.*

Proof. For contradiction, assume that $\bar{\kappa}$ is definably singularized over W_Θ. Let $\bar{M} \trianglelefteq W_\Theta$ such that $\bar{\kappa}$ is a cardinal in $\bar{M}$, but there is a good $\mathbf{\Sigma}_1^{(k)}(\bar{M})$-function singularizing $\bar{\kappa}$ for some $k < \omega$. Let n be the least such k. Note that $\rho_{\bar{M}}^{n+1} \leq \bar{\kappa} < \rho_{\bar{M}}^n$ in this case. We consider the canonical extension $\tilde{M} = \mathrm{Ult}_n(\bar{M}, \sigma \restriction (\bar{W}|\bar{\kappa}))$ of $\bar{M}$. Let $\tilde{\sigma} : \bar{M} \to \tilde{M}$ be the ultrapower map. The properties of the canonical extension (cf. Lemmas 3.6.3 - 3.6.7 of [16]), guarantee that $\tilde{\sigma}(\bar{\kappa}) = \kappa$, $\sup \tilde{\sigma}[\bar{\kappa}] = \kappa$, $\tilde{\sigma}$ is $\Sigma_0^{(n)}$-cofinal and $\rho_{\tilde{M}}^{n+1} \leq \kappa < \rho_{\tilde{M}}^n$. By the preservation properties of $\tilde{\sigma}$, κ is definably singularized over $\tilde{M}$ by a good $\mathbf{\Sigma}_1^{(n)}(\tilde{M})$-function. Next, we compare the phalanx $(W, \tilde{M}, \kappa)$ against W. The phalanx $(W, \tilde{M}, \kappa)$ is iterable by the methods of [9]. Let $(\bar{\mathcal{U}}, \mathcal{U})$ be the coiteration, where $\bar{\mathcal{U}}$ is on the phalanx side and suppose $\mathrm{lh}(\mathcal{U}) = \theta + 1 = \mathrm{lh}(\bar{\mathcal{U}})$.

Claim 2.7.1. The main branch of $\bar{\mathcal{U}}$ is above $\tilde{M}$.

Proof. Suppose not, so the root of the main branch of $\bar{\mathcal{U}}$ is W. Because W is a universal weasel, $M_\theta^{\mathcal{U}} \trianglerighteq M_\theta^{\bar{\mathcal{U}}}$ and there are no truncations along the main branch of $\bar{\mathcal{U}}$, so $M_\theta^{\bar{\mathcal{U}}}$ is a thick weasel. It follows that $M_\theta^{\mathcal{U}} = M_\theta^{\bar{\mathcal{U}}}$ and there are no truncations on the main branch of $\mathcal{U}$. Let M_θ denote this common final model. Let $\bar{\pi} = \pi_{-1,\theta}^{\bar{\mathcal{U}}} : W \to M_\theta$ be the iteration map on the $\bar{\mathcal{U}}$-side and $\pi = \pi_{0,\theta}^{\mathcal{U}} : W \to M_\theta$ be the iteration map on the $\mathcal{U}$-side. Let $\Delta = \{\alpha < \Omega \mid \pi(\alpha) = \alpha = \bar{\pi}(\alpha)\}$. By the usual reasoning, Δ is a thick class in W and M_θ. Let $\bar{E}$ be the first extender applied on the main branch of $\bar{\mathcal{U}}$ and E the first extender applied on the main branch of $\mathcal{U}$. We will show that $\bar{E}$ is compatible with E. Because W has the hull and definability property for all $\alpha < \Lambda$, focusing on the iteration on the $\mathcal{U}$-side $\mathrm{cr}(E)$ is the least ordinal $\alpha < \Lambda$ such that M_θ does not have the definability property at α. Similarly, focusing on the iteration on the $\bar{\mathcal{U}}$-side, $\mathrm{cr}(\bar{E})$ is the least ordinal $\alpha < \Lambda$ such that M_θ does not have the definability property at α. Hence, $\mathrm{cr}(\bar{E}) = \mathrm{cr}(E)$. Let ξ be this critical point. Let $\nu = \min\{\mathrm{lh}(E), \mathrm{lh}(\bar{E})\}$. Fix $a \in [\nu]^{<\omega}$ and $x \in \mathcal{P}([\xi]^{|a|})$ and notice because W has the hull and definability properties for all $\alpha < \Lambda$, $x = h_W(\vec{b})$ for some $\vec{b} \in \Delta$, where h_W is the uniformly definable Σ_1-Skolem function as defined over W. Hence, because critical points strictly increase along branches of an iteration, we have

$$x \in E_a \longleftrightarrow a \in \pi(x) \longleftrightarrow a \in \pi(h_W(\vec{b})) = h_{M_\theta}(\vec{b}) = \bar{\pi}(h_W(\vec{b}))$$
$$\longleftrightarrow a \in \bar{\pi}(x) \longleftrightarrow x \in \bar{E}_a$$

Thus, E and $\bar{E}$ are compatible. This gives us our contradiction and hence it must be that $\tilde{M}$ is the root of the main branch of $\bar{\mathcal{U}}$. □

Claim 2.7.2. $\tilde{M} \lhd W$

Proof. We consider three cases.

Case 1: $\tilde{M}$ moves in the coiteration, i.e. $M_\theta^{\bar{\mathcal{U}}} \neq \tilde{M}$.
Recall that $\rho_{\tilde{M}}^{n+1} \leq \kappa$ and the exchange ordinal of the phalanx is κ, so if E is the extender applied to $\tilde{M}$ along the main branch, then $\mathrm{cr}(E) \geq \kappa \geq \rho_{\tilde{M}}^{n+1}$. Thus, the resulting ultrapower $\mathrm{Ult}(\tilde{M}, E)$ will not be sound, and because there are no truncations along the main branch of $\bar{\mathcal{U}}$, $M_\theta^{\bar{\mathcal{U}}}$ will not be sound either. Because $M_\theta^{\bar{\mathcal{U}}}$ is not sound and there are no truncations along the main branch of $\bar{\mathcal{U}}$, we must have $M_\theta^{\bar{\mathcal{U}}} = M_\theta^{\mathcal{U}}$. Hence, there is a truncation along the main branch of $\mathcal{U}$.

Let $M_\theta = M_\theta^{\bar{\mathcal{U}}} = M_\theta^{\mathcal{U}}$, $M_{\zeta+1}^{\mathcal{U}}$ be the result of the last truncation, $\gamma = U(\zeta+1)$ and $M^* = M_{\zeta+1}^{*\mathcal{U}}$. Note that $\tilde{M}$ is sound above κ, and iterates above κ to M_θ, so

$$\rho_{\tilde{M}}^{n+1} = \rho_{M_\theta}^{n+1} \leq \kappa \quad \text{and} \quad \tilde{M} = \mathrm{core}_\kappa(M_\theta) \tag{2.1}$$

In particular, $\rho_{M_\theta}^{\omega} \leq \kappa$. First, we claim that $M_\gamma^{\mathcal{U}} = W$, so M^* is a truncate of W. If not, then let α be the length of the extender applied to W along the main branch and note that $\alpha > \kappa$. It follows that α is a cardinal in $M_\gamma^{\mathcal{U}}$. Because M^* is the last truncate along the main branch, $\rho_{M^*}^{\omega} = \rho_{M_\theta}^{\omega} \leq \kappa < \alpha$. So, α is not a cardinal in $M_\gamma^{\mathcal{U}}$ after all. Contradiction. Hence, $M_\gamma^{\mathcal{U}} = W$ and $M^* \lhd W$.

Let $E^* = E_\zeta$ which is the extender we apply to M^* and $\kappa^* = \kappa_\zeta$. Note that $\kappa^* \geq \kappa$ as κ is a cardinal in W, so we would not truncate if $\kappa^* < \kappa$. Consider $\rho^{n+1}_{M^*}$. First note that

$$\rho^{n+1}_{M^*} \geq \kappa \tag{2.2}$$

as κ is a cardinal in W. Moreover, $\rho^{n+1}_{M^*} \leq \kappa^*$ as otherwise $\rho^{n+1}_{M_\theta} > \kappa^* \geq \kappa$ but this contradicts Equation 2.1. Because the iteration from M^*-to-M_θ is above κ^*, we get $\rho^{n+1}_{M^*} = \rho^{n+1}_{M_\theta} = \rho^{n+1}_{\tilde{M}}$. Denote this common projectum by ρ^{n+1}. Hence, $\rho^{n+1} = \kappa$ by Equations 2.1 and 2.2.

It follows that $M^* = \mathrm{core}_\kappa(M_\theta)$ because M^* is sound above κ, projects to κ and the iteration from M^*-to-M_θ is above κ. Hence, by Equation 2.1 $M^* = \tilde{M}$. Thus, $\tilde{M} = M^* \lhd W$ which completes the proof of this case. $\square$ Case 1

Case 2: $\tilde{M}$ does not move in the coiteration, i.e. $M^{\bar{\mathcal{U}}}_\theta = \tilde{M}$ and there are no truncations on the main branch of $\mathcal{U}$.

In this case, $\tilde{M} \lhd M^{\mathcal{U}}_\theta$ and $M^{\mathcal{U}}_\theta$ is a weasel. Let F be the first extender used on the main branch of $\mathcal{U}$, and $\alpha = \mathrm{lh}(F)$. As above, $\alpha > \kappa$, and α will be a cardinal in $M^{\mathcal{U}}_\theta$. Because $\rho^{n+1}_{\tilde{M}} \leq \kappa$ and $\tilde{M} \lhd M^{\mathcal{U}}_\theta$, it must be that $\mathrm{ht}(\tilde{M}) < \alpha$. Hence, $\tilde{M} \lhd M^{\mathcal{U}}_\theta|\alpha$. This gives $\tilde{M} \lhd W$ as by the agreement in an iteration, we have $W|\alpha = M^{\mathcal{U}}_\theta|\alpha$. $\square$ Case 2

Case 3: $\tilde{M}$ does not move in the coiteration, i.e. $M^{\bar{\mathcal{U}}}_\theta = \tilde{M}$ and there is a truncation on the main branch of $\mathcal{U}$.

The argument for this case is very similar to Case 1, so we just summarize the main points. As in Case 1, let $M^{\mathcal{U}}_{\zeta+1}$ be the result of the last truncation on the main branch of $\mathcal{U}$, $\gamma = U(\zeta+1)$ and $M^* = M^{*\mathcal{U}}_{\zeta+1}$. Because there is a truncation on $\mathcal{U}$, $M^{\mathcal{U}}_\theta = M^{\bar{\mathcal{U}}}_\theta = \tilde{M}$, so $\rho^\omega_{\tilde{M}} = \rho^\omega_{M^*} \leq \kappa$. In the same way as in Case 1 it follows that $M^{\mathcal{U}}_\gamma = W$, so $M^* \lhd W$. Similarly, by looking at the critical point κ_ζ and the $(n{+}1\text{st})$-projecta we conclude that $\rho^{n+1}_{M^*} = \rho^{n+1}_{\tilde{M}} = \kappa$. It follows that $\mathrm{core}_\kappa(\tilde{M}) = M^*$ as M^* is sound above and projects to κ and there are no drops in model or degree after M^* along the main branch of $\mathcal{U}$. However, $\tilde{M}$ is also sound above and projects to κ, so $\tilde{M} = M^* \lhd W$, completing this case. $\square$ Case 3

Hence, in either case we arrive at $\tilde{M} \lhd W$ which completes the proof of the claim. $\square$

Recall, there is a good $\mathbf{\Sigma}_1^{(n)}(\tilde{M})$-function singularizing κ. By Claim 2.7.2 this function is an element of W, and hence κ is singular in W. This is a contradiction as κ is regular in W and completes the proof of the lemma. $\square$

Lemma 2.8. *The sequence $(\nu_i \mid i+1 \in (0,\Theta)_T)$ of natural lengths of extenders used along the main branch is cofinal in $\bar{\kappa}$. Moreover, the sequence of critical points $(\kappa_i \mid i+1 \in (0,\Theta)_T)$ of extenders used along the main branch of $\mathcal{T}$ are cofinal in $\bar{\kappa}$ as well.*

Proof. For contradiction, assume that the sequence of natural lengths of extenders is not cofinal in $\bar{\kappa}$ and let $\alpha^* < \bar{\kappa}$ such that $(\nu_i \mid i+1 \in (0,\Theta)_T)$ is bounded by α^*. Let $\varepsilon+1$ be past the last truncation on the main branch of $\mathcal{T}$ such that no drops in degree occur past ξ_ε. Because there is a truncation on the main branch of $\mathcal{T}$,

$$\rho^\omega_{W_{\varepsilon+1}} = \rho^\omega_{W^*_{\varepsilon+1}} \leq \kappa_\varepsilon < \nu_\varepsilon < \alpha^*$$

It follows that $\rho^\omega_{W_\Theta} = \rho^\omega_{W^*_{\varepsilon+1}} \leq \alpha^* < \bar{\kappa}$. Fix $n < \omega$ such that $\rho^{n+1}_{W_\Theta} \leq \alpha^* < \rho^n_{W_\Theta}$. By the properties of fine ultrapowers, W_Θ is sound above α^*, i.e. $\tilde{h}^{n+1}(\alpha^* \cup \{p_{W_\Theta}\}) = W_\Theta$ where $\tilde{h}^{n+1}$ is the good uniformly $\Sigma_1^{(n)}(W_\Theta)$-Skolem function. It follows that there is a good $\mathbf{\Sigma}_1^{(n)}(W_\Theta)$-function from a subset of α^* onto $\bar{\kappa}$. In other words, $\bar{\kappa}$ is definably singularized over W_Θ. This is a contradiction to Lemma 2.7.

For the moreover, just note that for each $i+1 < j+1 \in (0,\Theta)_T$

$$\kappa_i < \nu_i \leq \kappa_j$$

Hence, $\sup\{\kappa_i \mid i+1 \in (0,\Theta)_T\} = \sup\{\nu_i \mid i+1 \in (0,\Theta)_T\}$. $\square$

Now, we know that $\bar{\kappa}$ is not definably singularized over W_Θ (in particular $\bar{\kappa}$ is regular in

W_Θ) and that the sequence of critical points of extenders used along the main branch of $\mathcal{T}$ are cofinal in $\bar{\kappa}$.

Lemma 2.9. *The set $C_0 \subseteq (0,\Theta)_T$, defined by*

$$C_0 = \{\xi \in (0,\Theta)_T \mid \pi_{\xi,\Theta}(\mathrm{cr}(\pi_{\xi,\Theta})) = \bar{\kappa}\}$$

is cofinal in Θ.

Remark 2.10. Note that for $\xi \in C_0$, there is a unique $j+1 \in (0,\Theta)_T$ such that $\xi = \xi_j = T(j+1)$, and $\mathrm{cr}(\pi_{\xi,\Theta}) = \kappa_j$. Hence, C_0 could also be defined as

$$C_0 = \{\xi \in (0,\Theta)_T \mid \pi_{\xi_j,\Theta}(\kappa_j) = \bar{\kappa} \text{ where } j+1 \in (0,\Theta)_T \text{ is such that } \xi = \xi_j\}$$

Moreover, it is then easy to see that for $\xi_1 < \xi_2 \in C_0$, letting $\xi_1 = \xi_j$ and $\xi_2 = \xi_i$ where $i+1 <_T j+1$ we have

$$\pi_{\xi_j,\xi_i}(\kappa_j) = \kappa_i$$

Proof. Recall that $\varepsilon+1 \in (0,\Theta)_T$ is past any truncations and drops in degree along the main branch of $\mathcal{T}$, so for $i+1 \in (\varepsilon,\Theta)_T$, $W^*_{i+1} = W_{\xi_i}$. Let n be such that $\rho^{n+1}_{W_{\xi_\varepsilon}} \leq \kappa_\varepsilon < \rho^n_{W_{\xi_\varepsilon}}$. It follows that for $i+1 \in (\varepsilon,\Theta)_T$ we have $\rho^{n+1}_{W_{\xi_i}} \leq \kappa_i < \rho^n_{W_{\xi_i}}$. Let $k+1 \in (\varepsilon,\Theta)_T$ be least such that $\bar{\kappa} \in \mathrm{rng}(\pi_{\xi_k,\Theta})$. For $i+1 \in [k+1,\Theta)_T$, let $\bar{\kappa}_i$ be such that $\pi_{\xi_i,\Theta}(\bar{\kappa}_i) = \bar{\kappa}$. First, note that for any $i+1 \in [k+1,\Theta)_T$, $\bar{\kappa}_i \geq \kappa_i$. If this were not the case, then $\bar{\kappa} = \pi_{\xi_i,\Theta}(\bar{\kappa}_i) < \kappa_i$ which is impossible.

Claim 2.10.1. On a cofinal set of $i+1 \in [k+1,\Theta)_T$, $\bar{\kappa}_i = \kappa_i$.

Note, it suffices to prove the claim as the claim proves that C_0 is cofinal in Θ. As we mentioned in the remark, if $\xi \in C_0$, then there is a unique $i+1 \in (0,\Theta)_T$ such that $\xi = \xi_i$ and $\mathrm{cr}(\pi_{\xi_i,\Theta}) = \kappa_i$. Hence, if $i+1$ satisfies the claim, then $\mathrm{cr}(\pi_{\xi_i,\Theta}) = \bar{\kappa}_i$, so $\pi_{\xi_i,\Theta}(\mathrm{cr}(\pi_{\xi_i,\Theta})) = \bar{\kappa}$.

Proof of claim. Assume for a contradiction that the claim fails. Then, there is $i_0 \in [k{+}1, \Theta)_T$ such that for all $i+1 \in [i_0, \Theta)_T$ we have $\bar{\kappa}_i > \kappa_i$. Fix an $i+1 \in [i_0, \Theta)_T$. We first claim that, $\bar{\kappa}_i \leq \rho^n_{W_{\xi_i}}$. If not, then by the preservation properties of the ultrapower map, we would have $\bar{\kappa} = \pi_{\xi_i,\Theta}(\bar{\kappa}_i) > \rho^n_{W_\Theta}$. But $\bar{\kappa} = \sup\{\kappa_j \mid j+1 \in [k+1, \Theta)\}$ and each such κ_j satisfies $\kappa_j < \rho^n_{W_{\xi_j}}$. Hence, in the direct limit we have $\bar{\kappa} \leq \rho^n_{W_\Theta}$. Thus, we must have $\bar{\kappa}_i \leq \rho^n_{W_{\xi_i}}$.

Next we show that $\bar{\kappa}_i$ is mapped cofinally to $\bar{\kappa}_{i'}$ where $i'{+}1$ is the immediate successor of $i{+}1$ along the main branch of $\mathcal{T}$. That is, $i'{+}1 \in (i_0, \Theta)_T$ is such that $T(i'{+}1) = \xi_{i'} = i{+}1$. Note then, that the map $\pi_{\xi_i,\xi_{i'}}$ is just the ultrapower map $\pi^*_{i+1} : W_{\xi_i} \to W_{i+1} = W_{\xi_j}$. If $\bar{\kappa}_i = \rho^n_{W_{\xi_i}}$ this follows from the fact that the ultrapower map $\pi_{\xi_i,\xi_{i'}}$ is $\Sigma_0^{(n)}$-cofinal. If $\bar{\kappa}_i < \rho^n_{W_{\xi_i}}$ then any element $x \in W_{i+1}$ such that $x < \pi_{\xi_i,\xi_{i'}}(\bar{\kappa}_i) = \bar{\kappa}_{i'}$ is of the form $\pi_{\xi_i,\xi_{i'}}(f)(a)$ for some function $f : [\kappa_i]^{|a|} \to \bar{\kappa}_i \in W_{\xi_i}$. Note that, $\bar{\kappa}_i$ is regular in W_{ξ_i} because its image $\bar{\kappa}$ is regular in W_Θ. Because $\bar{\kappa}_i$ is regular in W_{ξ_i} and $\bar{\kappa}_i > \kappa_i$ by assumption, the function f is bounded by $\alpha < \bar{\kappa}_i$ so $\pi_{\xi_i,\xi_{i'}}(f)(a) < \pi_{\xi_i,\xi_{i'}}(\alpha)$. This shows $\bar{\kappa}_i$ is mapped cofinally to $\bar{\kappa}_{i'}$ via the ultrapower map.

Inductively we can show that for $j+1 < j'+1 \in (i_0, \Theta)_T$ $\bar{\kappa}_j$ is mapped cofinally to $\bar{\kappa}_{j'}$ via the iteration map $\pi_{\xi_j,\xi_{j'}}$ using the assumption that $\bar{\kappa}_l > \kappa_l$ for all $l \in (i_0, \Theta)_T$. By the properties of the direct limit, it follows that $\bar{\kappa}_i$ is mapped cofinally to $\bar{\kappa}$ via $\pi_{\xi_i,\Theta}$. Consider the hull $\tilde{h}^{n+1}_{W_{\xi_i}}(\kappa_i \cup \{p_M\})$. Note that W_{ξ_i} is sound above κ_i as it is past the last truncation and the lengths of extenders used along the branch to W_{ξ_i} are less than κ_i. So, this hull is all of W_{ξ_i}. In particular, this gives us a good $\mathbf{\Sigma}_1^{(n)}(W_{\xi_i})$-function from a subset of κ_i onto $\bar{\kappa}_i$ i.e. $\bar{\kappa}_i$ is definably singularized over W_{ξ_i}. It follows that $\bar{\kappa}$ will be definably singularized over W_Θ by the preservation properties of the ultrapower map $\pi_{\xi_i,\Theta}$ and the fact that $\bar{\kappa}_i$ is mapped cofinally to $\bar{\kappa}$. This contradicts Lemma 2.7 and completes the proof of the claim. □

This completes the proof of the lemma. As mentioned after the statement of the claim, the claim directly implies the desired result. □

Fix a cofinal set $C_0 \subseteq (0, \Theta)_T$ satisfying Lemma 2.9. Recall that κ has cofinality ω, and since $\sup\{\kappa_i \mid i+1 \in C_0\} = \bar{\kappa}$, it follows that $\mathrm{cf}(\Theta) = \omega$ as well. Hence, we can fix a cofinal set $C_1 \subseteq C_0$ such that $\mathrm{otp}(C_1) = \omega$. Without loss of generality, assume $C_1 \subseteq (\varepsilon, \Theta)_T$ where $\varepsilon + 1 \in (0, \Theta)_T$ is past any truncations or drops in degree. Recall that we have a countably closed structure H such that $\bar{\kappa} \in H$, and so $\Theta \in H$ as well. Hence, ${}^{\omega}\Theta \subseteq H$, so $C_1 \in H$. In the following we consider two cases

Case 1 On a tail-end of $\xi_i \in C_1$ the extender E_i is essentially a measure of order zero, i.e. the only generator for E_i is κ_i and $o_*^{W_i}(E_i) = 0$

Case 2 Case 1 does not happen i.e. for cofinally many $\xi_i \in C_1$ E_i is not essentially a measure of order zero.

In an abuse of language we will call an extender "a measure" to mean that the extender's sole generator is its critical point. We will also abuse notation and consider measures of the form $E_{\{\mathrm{cr}(E)\}}$ where E is an extender as measures on $\mathcal{P}(\mathrm{cr}(E))$ instead of on $\mathcal{P}([\mathrm{cr}(E)]^1)$ to simplify the notation. In either of the two cases above, we show that inside H we can build a normal measure $\bar{F}$ on $\mathcal{P}(\bar{\kappa}) \cap \bar{W}$. We will then show that $\bar{F}$ is compatible with an extender on the $\bar{W}$-sequence and from there complete the proof of Theorem 2.3.

We now deal with Case 1. Let $D \subseteq C_1$ be a tail-end of C_1 such that for all $\xi_i \in D$, E_i is a measure of order zero. Notice, for $\xi_i \in D$ because E_i is a measure of order zero, E_i is applied to W_i. In other words, $\xi_i = i$ for $\xi_i \in D$ and the ultrapower map π^*_{i+1} is just $\pi_{i,i+1}$. Let F be the common value of $\pi_{i,\Theta}(E_i)$ for $i \in D$.

Lemma 2.11. *For $x \in \mathcal{P}(\bar{\kappa}) \cap W_\Theta$,*

$$x \in F_{\{\bar{\kappa}\}} \longleftrightarrow (\exists j \in D)\ \kappa_i \in x \text{ for all } i \in D - j$$

Proof. First note that if $x \in \mathcal{P}(\kappa) \cap W_\Theta$ and $j \in D$ is large enough such that $x \in \mathrm{rng}(\pi_{j,\Theta})$, then letting $\bar{x}$ be $\pi^{-1}_{j,\Theta}(x)$ we have

$$x \in F_{\{\bar{\kappa}\}} \longleftrightarrow \bar{x} \in E_j \longleftrightarrow \kappa_j \in \pi_{j,j+1}(\bar{x}) \longleftrightarrow \kappa_j \in \pi_{j,\Theta}(\bar{x}) = x \tag{2.3}$$

Then, if $i \in D$ is such that $j < i$ then $\pi_{j,i}(\kappa_j) = \kappa_i$ and $\pi_{j,i}(E_j) = E_i$. Hence, for any $i > j$ with $i \in D$ we have

$$x \in F_{\{\bar{\kappa}\}} \longleftrightarrow \bar{x} \in E_j \longleftrightarrow \pi_{j,i}(\bar{x}) \in E_i \longleftrightarrow \kappa_i \in \pi_{j,i+1}(\bar{x}) \longleftrightarrow \kappa_i \in \pi_{j,\Theta}(\bar{x}) = x \tag{2.4}$$

Thus, for any $i \in D$ with $i > j$ we have shown

$$x \in F_{\{\bar{\kappa}\}} \longleftrightarrow \kappa_i \in x \tag{2.5}$$

which proves the lemma. $\square$

Inside H, we define a measure $\bar{F}$ on $\mathcal{P}(\bar{\kappa}) \cap \bar{W}$ following the equivalence in Lemma 2.11. More precisely, for $y \in \mathcal{P}(\bar{\kappa}) \cap \bar{W}$

$$y \in \bar{F} \longleftrightarrow (\exists j \in D)\ \kappa_i \in x \text{ for all } i \in D - j \tag{2.6}$$

Remark 2.12. Note that D and each κ_i is in H, so that we can indeed build $\bar{F}$ inside H.

Next, we would like to show that $\bar{F}$ is measure on $\mathcal{P}(\bar{\kappa}) \cap \bar{W}$, i.e. it is a normal κ-complete ultrafilter. To do this, we first show that $\bar{W}$ and W_Θ agree up to $\bar{\kappa}^{+\bar{W}}$.

Lemma 2.13. $\mathcal{P}(\bar{\kappa}) \cap \bar{W} \subseteq W_\Theta$

Proof. We already know that $\bar{W}|\bar{\kappa} \lhd W_\Theta$. To show that $\mathcal{P}(\bar{\kappa}) \cap \bar{W}$ is also contained in W_Θ, we consider the coiteration of $\bar{W}|\bar{\tau}$ against W, where $\bar{\tau} = \bar{\kappa}^{+\bar{W}}$. Let $(\bar{\mathcal{T}}', \mathcal{T}')$ denote this coiteration and Θ' be its length. As in Fact 2.5, we know that $\bar{W}|\bar{\tau}$ does not move in this coiteration and that $\bar{W}|\bar{\tau} \lhd M^{\mathcal{T}'}_{\Theta'}$. Moreover, it is the case that $\mathcal{T}'$ extends $\mathcal{T}$. Note, that if $\mathcal{T} = \mathcal{T}'$, then $M^{\mathcal{T}'}_{\Theta'} = W_\Theta$, so there is nothing to show. So assume that $\mathcal{T}'$ strictly extends $\mathcal{T}$ and so $M^{\mathcal{T}'}_{\Theta} = W_\Theta$. Let $\alpha = \mathrm{lh}(E^{\mathcal{T}'}_{\Theta})$ and note that $\alpha > \bar{\kappa}$ as $\bar{\kappa}$ is a cardinal in W_Θ. Moreover, α is a cardinal in $M^{\mathcal{T}'}_{\Theta'}$ and $M^{\mathcal{T}'}_{\Theta'}|\alpha = W_\Theta|\alpha$. Because α is a cardinal in $M^{\mathcal{T}'}_{\Theta'}$, and $\alpha > \bar{\kappa}$ it follows that $\alpha \geq \bar{\tau}$. Hence, $\bar{W}|\bar{\tau} \unlhd M^{\mathcal{T}'}_{\Theta'}|\alpha = W_\Theta|\alpha$. Thus, $\mathcal{P}(\bar{\kappa}) \cap \bar{W} \subseteq W_\Theta$. □

Lemma 2.14. *$\bar{F}$ is a normal κ-complete ultrafilter on $\mathcal{P}(\bar{\kappa}) \cap \bar{W}$.*

Proof. This is essentially by construction as by Lemma **??** $\mathcal{P}(\bar{\kappa}) \cap \bar{W} \subseteq \mathcal{P}(\bar{\kappa}) \cap W_\Theta$ so $\bar{F}$ agrees with $F_{\{\bar{\kappa}\}}$ on $\mathcal{P}(\bar{\kappa}) \cap \bar{W}$ and $F_{\{\bar{\kappa}\}}$ is a measure as F is on the W_Θ-sequence. For normality, we just note that if $x \in \bar{F}$ and $f : x \to \bar{\kappa} \in \bar{W}$ is regressive, then each set $a_\xi = f^{-1}[\xi] \in \mathcal{P}(\bar{\kappa}) \cap \bar{W}$. Note that $f \in \bar{W}|\tau$ by acceptability, so that $f \in W_\Theta$. Then, $F_{\{\bar{\kappa}\}}$ must concentrate on one a_ξ and so this $a_\xi \in \bar{F}$. □

We have now constructed a normal measure $\bar{F}$ on $\mathcal{P}(\bar{\kappa}) \cap \bar{W}$ inside H under the assumption we are in Case 1. We now consider Case 2 and show we can similarly build a measure $\bar{F} \in H$.

Let $C \subseteq C_1$ be cofinal in C_1 and such that for all $\xi_i \in C$, $o^{W_i}_*(E_i) > 0$. Recall that $C_1 \in H$, $\mathrm{otp}(C_1) = \omega$, and that $\sup\{\kappa_i \mid i+1 \in C_1\} = \bar{\kappa}$, so the same facts hold of C. For each $\xi_i \in C$, let F_i be the measure of order zero on the W_i-sequence with critical point κ_i (equivalently, F_i is the extender on the W_i-sequence with least index and critical point κ_i).

Lemma 2.15. *For each $\xi_i \in C$, F_i is on the sequence of the model we apply E_i to, that is F_i is on the $\widehat{W_{\xi_i}}$-sequence.*

Proof. To see this, fix a $\xi_i \in C$ and let $j = \xi_i$. The claim is clear if $j = i$, so assume $j < i$. Then, $\kappa_i < \nu_j$ so that $\tau_i \leq \alpha_j$ as α_j is a cardinal in W_i. Note that $W^*_{i+1} = W_j$ as we are beyond any truncations, and that we apply E_i to $\widehat{W_j}$. Let $\alpha = \mathrm{lh}(F_i)$ and note that $\tau_i < \alpha < \tau_i^{+W_i}$ as F_i is just the trivial completion of a measure.

First, consider the case that $\tau_i < \alpha_j$. It follows that $\tau_i^{+W_i} \leq \alpha_j$ as α_j is a cardinal in W_i. Moreover, $W_i|\alpha_j = W_j|\alpha_j$, so $\alpha < \tau_i^{+W_i} \leq \alpha_j$ and we get the stronger conclusion that F_i is on the W_j-sequence.

Next, consider the case that $\tau_i = \alpha_j$. Note that this implies that E_j is the top extender of W_j. If E_j was not the top extender than α_j would not be a cardinal in W_j. However, we know $\tau_i = \alpha_j$ is a cardinal in W_j as we do not truncate. Recall, $\widehat{W_j} = \mathrm{Ult}^0(W_j|\tau_j, E_j)$ which means $\widehat{W_j}$ and $W_{j+1} = \mathrm{Ult}^*(W^*_{j+1}, E_j)$ agree up to their common value for $\pi_{E_j}(\kappa_j) = \pi^*_{j+1}(\kappa_j)$. Moreover, $\pi_{E_j}(\kappa_j) > \alpha_j = \tau_i$ and recall that $\alpha < \tau_i^{+W_i}$. If $W_{j+1} = W_i$, then clearly $\alpha < \pi_{E_j}(\kappa_j)$, so F_i is on the $\widehat{W_j}$-sequence. Otherwise, $j + 1 <_T i$. Let $\alpha^* = \min\{\pi_{E_j}(\kappa_j), \alpha_{j+1}\}$. Then, α^* is a cardinal in W_i and $\alpha^* > \alpha_j = \tau_i$, so $\alpha^* \geq \tau_i^{+W_i}$. Hence, $\alpha^* > \alpha$. Moreover, we have the agreement $\widehat{W_j}|\alpha^* = W_{j+1}|\alpha^* = W_i|\alpha^*$ and it follows that F_i is on the $\widehat{W_j}$-sequence. □

So, for each $\xi_i \in C$, F_i is on the $\widehat{W_{\xi_i}}$-sequence. As F_i is the extender with smallest index on the W_i-sequence with critical point κ_i, it follows that for $\xi_i < \xi_j \in C$, $F_j = \pi_{\xi_i,\xi_j}(F_i)$. Releasing the previous notation, let F be the common value for $\pi_{\xi_i,\Theta}(F_i)$ for $\xi_i \in C$.

Lemma 2.16. *For $x \in \mathcal{P}(\kappa) \cap W_\Theta$,*

$$x \in (F)_{\{\bar{\kappa}\}} \longleftrightarrow (\exists j \in C)\ x \cap \kappa_i (F_i)_{\{\kappa_i\}} \text{ for all } \xi_i \in C - j$$

Proof. First note that if $x \in \mathcal{P}(\kappa) \cap W_\Theta$ and $\xi_i \in C$ is large enough such that $x \in \mathrm{rng}(\pi_{\xi_i,\Theta})$, then letting $\bar{x} = \pi^{-1}_{\xi_i,\Theta}(x)$ we have

$$x \in (F)_{\{\bar{\kappa}\}} \longleftrightarrow \bar{x} \in (F_i)_{\{\kappa_i\}} \longleftrightarrow x \cap \kappa_i \in F_{i,\{\kappa_i\}} \tag{2.7}$$

Where the last equivalence holds because $\mathrm{cr}(\pi_{\xi_i,\Theta}) = \kappa_i$, so $\bar{x} = \pi_{\xi_i,\Theta}(\bar{x}) \cap \kappa_i = x \cap \kappa_i$. As

Equation 2.7 holds for all large enough $\xi_i \in C$, it follows that for $x \in \mathcal{P}(\kappa) \cap W_\Theta$

$$x \in (F)_{\{\bar{\kappa}\}} \longleftrightarrow (\exists j \in C)\ x \cap \kappa_i \in (F_i)_{\{\kappa_i\}} \text{ for all } \xi_i \in C - j \tag{2.8}$$

□

As in Case 1, we use this equivalence to define a measure $\bar{F} \in H$ on $\mathcal{P}(\bar{\kappa}) \cap \bar{W}$. Note that for $\xi_i \in C$, F_i is on the $\bar{W}$-sequence by coherence. Hence, inside H we have access to C and F_i. So, given $y \in \mathcal{P}(\bar{\kappa}) \cap \bar{W}$ we set

$$y \in \bar{F} \longleftrightarrow (\exists j \in C)\ y \cap \kappa_i \in (F_i)_{\{\kappa_i\}} \text{ for all } \xi_i \in C - j \tag{2.9}$$

Similarly to Case 1, because $\bar{F}$ agrees with F on $\mathcal{P}(\bar{\kappa}) \cap \bar{W}|\bar{\kappa}$, it is not hard to see

Lemma 2.17. *$\bar{F}$ is a normal κ-complete ultrafilter on $\mathcal{P}(\bar{\kappa}) \cap \bar{W}$.*

Proof. Same as Lemma 2.14. □

This completes the construction of $\bar{F}$ in Case 2. In what follows, the exact definition of $\bar{F}$ does not matter. The important fact is that in either Case 1 or Case 2 we have constructed a normal measure $\bar{F}$ on $\mathcal{P}(\bar{\kappa}) \cap \bar{W}$ inside H.

Our next goal is to show that $\bar{F}$ is compatible with some extender on the $\bar{W}$-sequence. For the next few lemmas, we will work inside of H. Note that, inside of H, $\bar{W}$ is a soundness witness for $\mathsf{K}^H|\bar{\Lambda}$ where $\bar{\Lambda} = \sigma^{-1}(\Lambda)$. Hence, all the facts of soundness witnesses hold in H for $\bar{W}$. Form the ultrapower $\bar{W}' = \mathrm{Ult}(\bar{W}, \bar{F})$ of $\bar{W}$ by $\bar{F}$ and let $\bar{\pi} : \bar{W} \to \bar{W}'$ denote the ultrapower map.

Lemma 2.18. *$\bar{W}'$ is fully iterable*

Proof. By Corollary 15 of [1] it suffices to show that $\mathrm{Ult}(\bar{W}, \bar{F})$ is well founded. □

Lemma 2.19. *The ultrapower* $\mathrm{Ult}(\bar{W}, \bar{F})$ *is well-founded.*

Proof. Suppose this was not the case and let $(f_i \mid i < \omega) \in H$ witness the ill-foundedness where each $f_i : \bar{\kappa} \to \bar{W} \in \bar{W}$. Inside H, construct an elementary substructure $\bar{X} \prec H_{\bar{\Lambda}}$ such that:

- $(f_i \mid i < \omega), \bar{W}, \bar{\kappa}, \bar{\tau}, \bar{F} \in \bar{X}$
- $\bar{\kappa} \subseteq \bar{X}$
- $\mathrm{card}(\bar{X}) = \mathrm{card}(\bar{\kappa})$

Collapse $\bar{X}$ to obtain a map $\bar{\sigma}_0 : \bar{H} \to \bar{X}$. Let $(\bar{W}_0, \bar{\tau}_0) = \bar{\sigma}_0^{-1}(\bar{W}, \bar{\tau})$ and $\bar{\sigma} = \bar{\sigma}_0 \restriction \bar{W}_0 : \bar{W}_0 \to \bar{W}$. Let $\bar{\tau}^* = \sup \bar{\sigma}[\bar{\tau}_0]$ and let $\tilde{\sigma} : \bar{W}_0 \to \bar{W}^*$ be the canonical extension of $\bar{\sigma}_0 \restriction \bar{W}_0 | \bar{\tau}_0$. By the interpolation lemma, there is a map $\sigma^* : \bar{W}^* \to \bar{W}$ such that $\mathrm{cr}(\sigma^*) = \bar{\tau}^*$ and $\sigma^*(\bar{\tau}^*) = \bar{\tau}$.

For $i < \omega$, let $\bar{f}_i^* = \tilde{\sigma}(\bar{\sigma}^{-1}(f_i))$. If we let $\bar{F}^*$ be the restriction of $\bar{F}$ to $\mathcal{P}(\bar{\kappa}) \cap \bar{W}^*$, then the functions $(\bar{f}_i^* \mid i < \omega)$ witness that $\mathrm{Ult}(\bar{W}^*, \bar{F}^*)$ is ill-founded. More precisely, letting z_i denote the set

$$z_i = \{\alpha < \bar{\kappa} \mid \bar{f}_{i+1}^*(\alpha) \in \bar{f}_i^*(\alpha)\} \tag{2.10}$$

we have $z_i \in \bar{W}^*$ and $z_i \in \bar{F}^* \subseteq \bar{F}$.

Note that we can embed the phalanx $(\bar{W}||\bar{\Lambda}, \bar{W}^*, \bar{\tau}^*)$ into $\bar{W}$ so it is iterable. Let $(\bar{\mathcal{V}}, \mathcal{V})$ be the coiteration of $(\bar{W}||\bar{\Lambda}, \bar{W}^*, \bar{\tau}^*)$ against $\bar{W}||\bar{\Lambda}$. By Theorem 3.4 of [12] applied inside H, $\bar{W}||\bar{\Lambda}$ is universal for all mice of height at most $\bar{\Lambda}$, so $\bar{W}||\bar{\Lambda}$ will win this coiteration. Moreover, by the standard arguments using the hull and definability property, $\bar{W}^*$ will be the main branch on the phalanx side. Let Q^* be the last model of $\bar{\mathcal{V}}$, Q be the last model of $\mathcal{V}$, and $\pi^* : \bar{W}^* \to Q^*$, $\pi : \bar{W}||\bar{\Lambda} \to Q$ the iteration maps. So, $Q^* \trianglelefteq Q$, $\mathrm{cr}(\pi^*) \geq \bar{\tau}^*$, and $\mathcal{P}(\bar{\kappa}) \cap \bar{W}^* = \mathcal{P}(\bar{\kappa}) \cap Q^*$.

For $i < \omega$, let $f_i^* = \pi^*(\bar{f}_i^*)$ and note that $\mathrm{dom}(f_i^*) = \bar{\kappa}$ as $\mathrm{cr}(\pi^*) \geq \bar{\tau}^*$. Recall that we have sets $z_i \in \mathcal{P}(\bar{\kappa}) \cap \bar{W}^*$ for $i < \omega$ defined in Equation 2.10 such that each $z_i \in \bar{F}^* \subseteq \bar{F}$. Because $\mathrm{cr}(\pi^*) \geq \bar{\tau}^*$, $\pi^*(z_i) = z_i$ and if $\alpha \in z_i$, then $f_{i+1}^*(\alpha) \in f_i^*(\alpha)$. It follows that the functions $(f_i^* \mid i < \omega)$ witness the ill-foundedness of $\mathrm{Ult}(Q^*, \bar{F}^*)$ and thus also the ill-foundedness of $\mathrm{Ult}(Q, \bar{F})$. The ultimate contradiction we arrive out will come from this fact after we copy the tree $\mathcal{V}$ to a tree $\mathcal{V}'$ on $\mathrm{Ult}(\bar{W}||\bar{\Lambda}, \bar{F})$. First, we claim

Claim 2.19.1. $\mathrm{Ult}(\bar{W}||\bar{\Lambda}, \bar{F})$ is well-founded

For now, let us assume this claim and complete the proof of the lemma. So, $\bar{W}' = \mathrm{Ult}(\bar{W}||\bar{\Lambda}, \bar{F})$ is well-founded. Let $\mathcal{V}' = \pi_{\bar{F}}\mathcal{V}$ be the copy of the tree $\mathcal{V}$ onto $\bar{W}'$ using the ultrapower map $\pi_{\bar{F}} : \bar{W}||\bar{\Lambda} \to \bar{W}'$ as the initial copy map. Let $\sigma' : Q \to Q'$ be the final copy map between the final models of $\mathcal{V}$ and $\mathcal{V}'$. Note that because $\bar{W}||\bar{\Lambda}$ and $\bar{W}^*$ agree up to $\bar{\tau}^*$, the first iteration index on the tree $\mathcal{V}$ will be at least $\bar{\tau}^*$. So, the agreement in the copy construction guarantees that

$$\pi_{\bar{F}} \restriction \bar{\tau}^* = \sigma' \restriction \bar{\tau}^* \tag{2.11}$$

Let us now trace our steps back to arrive at the final contradiction. Recall that the functions $(f_i^* \mid i < \omega)$ witnessed the ill-foundedness of $\mathrm{Ult}(Q, \bar{F})$ using the sets z_i. For $i < \omega$, $z_i \in \bar{F}$, so by Equation 2.11 we have

$$z_i \in \bar{F} \longleftrightarrow \bar{\kappa} \in \pi_{\bar{F}}(z_i) \longleftrightarrow \bar{\kappa} \in \sigma'(z_i) \tag{2.12}$$

It follows that the sequence $(\sigma'(f_i^*)(\bar{\kappa}) \mid i < \omega)$ constitutes an infinite $\in$-decreasing sequence in Q'. However, Q' was an iterate of $\bar{W}'$ so it is well-founded. Contradiction. Now, we prove Claim 2.19.1.

Proof of Claim 2.19.1. Let $(\mathcal{T}^*, \bar{\mathcal{T}}^*)$ be the coiteration of W against $\bar{W}||\bar{\Lambda}$. Note, as in

Lemma 2.13 $\mathcal{T}^*$ extends the iteration tree $\mathcal{T}$, and $\bar{W}||\bar{\Lambda}$ does not move. Thus, letting Θ^*+1 be the length of this coiteration, $\bar{W}||\bar{\Lambda} \trianglelefteq M^{\mathcal{T}^*}_{\Theta^*}$. We are going to "copy" the tree $\mathcal{T}^*$ to a tree $\tilde{\mathcal{T}}$ on W as follows:

- $\tilde{\mathcal{T}} \restriction \Theta+1 = \mathcal{T}^* \restriction \Theta+1 = \mathcal{T} \restriction \Theta+1$
- $M^{\tilde{\mathcal{T}}}_{\Theta+1} = \mathrm{Ult}^*(M^{\tilde{\mathcal{T}}}_{\Theta}, F)$ (recall F is on the $W_\Theta = M^{\tilde{\mathcal{T}}}_{\Theta}$-sequence)
- For $\alpha < \Theta$, the copy map $\sigma_\alpha : M^{\mathcal{T}^*}_{\alpha} \to M^{\tilde{\mathcal{T}}}_{\alpha} = \mathrm{id}$
- The copy map $\sigma_\Theta : M^{\mathcal{T}^*}_{\Theta} \to M^{\tilde{\mathcal{T}}}_{\Theta+1}$ is the ultrapower map determined by F
- There is no copy map with $M^{\tilde{\mathcal{T}}}_{\Theta}$ as its target model
- For $\alpha > \Theta$, the copy maps are given by the copy construction

Note, that the tree $\tilde{\mathcal{T}}$ as defined is normal, as $\mathrm{cr}(F) = \bar{\kappa} > \nu^{\tilde{\mathcal{T}}}_i$ for any $i < \Theta$. The following diagram summarizes the construction:

$$\begin{tikzcd}
M^{\tilde{\mathcal{T}}}_0 \arrow[r] & \cdots M^{\tilde{\mathcal{T}}}_\alpha \arrow[r] & \cdots M^{\tilde{\mathcal{T}}}_\Theta \arrow[r, "F"] & M^{\tilde{\mathcal{T}}}_{\Theta+1} \arrow[r] & M^{\tilde{\mathcal{T}}}_{\Theta+2} \arrow[r] & \cdots \arrow[r] & M^{\tilde{\mathcal{T}}}_{\Theta^*} \\
M^{\mathcal{T}^*}_0 \arrow[u, "\sigma_0=\mathrm{id}"] \arrow[r] & \cdots M^{\mathcal{T}^*}_\alpha \arrow[u, "\sigma_\alpha=\mathrm{id}"] \arrow[r] & \cdots M^{\mathcal{T}^*}_\Theta \arrow[ur, "\sigma_\Theta"] \arrow[r] & M^{\mathcal{T}^*}_{\Theta+1} \arrow[ur, "\sigma_{\Theta+1}"] \arrow[r] & \cdots \arrow[r] & M^{\mathcal{T}^*}_{\Theta^*} \arrow[ur, "\sigma_{\Theta^*}"]
\end{tikzcd}$$

By Lemma 2.13, $\bar{W}||\bar{\tau} \trianglelefteq W_\Theta = M^{\mathcal{T}^*}_{\Theta}$. It follows that $\alpha^{\mathcal{T}^*}_{\Theta} \geq \bar{\tau}$. Hence, the agreement in the copy construction, guarantees

$$\sigma_\Theta \restriction \bar{\tau} = \sigma_{\Theta+1} \restriction \bar{\tau} = \sigma_{\Theta^*} \restriction \bar{\tau}$$

Note that σ_Θ was defined to be the ultrapower map π_F, so for $x \in \mathcal{P}(\bar{\kappa}) \cap \bar{W} = \mathcal{P}(\bar{\kappa}) \cap \bar{W}||\bar{\tau}$

$$x \in \bar{F} \longleftrightarrow x \in F_{\{\bar{\kappa}\}} \longleftrightarrow \bar{\kappa} \in \pi_F(x) \longleftrightarrow \bar{\kappa} \in \sigma_\Theta(x) \longleftrightarrow \bar{\kappa} \in \sigma_{\Theta^*}(x) \tag{2.13}$$

Recall that $\bar{W}||\bar{\Lambda} \trianglelefteq M^{\mathcal{T}^*}_{\Theta^*}$ and let $\tilde{W} = \mathrm{Ult}(\bar{W}||\bar{\Lambda}, \bar{F})$. Using Equation 2.13, we can define a factor map $k : \tilde{W} \to M^{\tilde{\mathcal{T}}}_{\Theta^*}$ in the usual way. By Łoś's theorem, this map will be Σ_0-preserving. Because $M^{\tilde{\mathcal{T}}}_{\Theta^*}$ is a normal iterate of W, it is well-founded. Thus, $\tilde{W} = \mathrm{Ult}(\bar{W}||\bar{\Lambda}, \bar{F})$ is well-founded which proves the claim. □

□

Now, we compare the phalanx $(\bar{W}, \bar{W}', \bar{\kappa})$ against $\bar{W}$. As the phalanx $(\bar{W}, \bar{W}', \bar{\kappa})$ can be embedded into $\bar{W}'$, this phalanx is iterable. Let $\bar{\mathcal{S}}$, $\mathcal{S}$ be the coiteration of $(\bar{W}, \bar{W}', \bar{\kappa})$ against $\bar{W}$.

Lemma 2.20. *In this coiteration, the models iterate to a common model $\bar{W}^*$, there are no truncations on the main branch of either tree, and the main branch of $\bar{\mathcal{S}}$ is above $\bar{W}'$.*

Proof. Let $\bar{\Theta} + 1$ be the length of this coiteration. Note that $\bar{W}$ and $\bar{W}'$ are both weasels (they have height $\bar{\Omega}$) and because $\bar{W}$ is a soundness witness, it is universal. Hence, $M^{\bar{\mathcal{S}}}_{\bar{\Theta}} \trianglelefteq M^{\mathcal{S}}_{\bar{\Theta}}$ and there are no truncations along the main branch of $\bar{\mathcal{S}}$.

We claim that $M^{\bar{\mathcal{S}}}_{\bar{\Theta}} = M^{\mathcal{S}}_{\bar{\Theta}}$. For contradiction, suppose not and let $Q = M^{\bar{\mathcal{S}}}_{\bar{\Theta}}$ and $P = M^{\mathcal{S}}_{\bar{\Theta}}$, so $Q \triangleleft P$. It follows that $\bar{\Theta} = \bar{\Omega}$. We remark that the assumption $Q \triangleleft P$ also incorporates into it the possibility that there is a truncation on the main branch of $\mathcal{S}$. This is because after truncating the resulting model will be a mouse and the only way a mouse can beat a weasel in a coiteration with no truncations on the weasel side is if it iterates strictly past it.

So, if there is a truncation on the main branch of $\mathcal{T}$, let $\eta + 1$ be the result of the last truncation and let $\xi_\eta = \gamma$. If there is no truncation, let $\gamma = 0$. In either case, the iteration map $\pi^{\mathcal{S}}_{\gamma,\bar{\Theta}}$ is total.

As, there is no truncation on the main branch of $\mathcal{S}$, Q is a thick weasel inside H. It follows that $\mathrm{ht}(P) > \bar{\Omega}$, so there is $\alpha_{\mathcal{S}}$ such that for all $\xi \in (\gamma, \bar{\Theta})_S$

$$\pi^{\mathcal{S}}_{\gamma,\xi}(\alpha_{\mathcal{S}}) \geq \mathrm{cr}(\pi^{\mathcal{S}}_{\xi,\bar{\Omega}})$$

Using this, we can find a club $Z \subseteq [\gamma, \bar{\Omega}]_S \cap \mathrm{Lim}$ such that for all $\eta < \eta' \in Z$

$$\pi^{\mathcal{S}}_{\eta,\eta'}(\mathrm{cr}(\pi^{\mathcal{S}}_{\eta,\bar{\Omega}})) = \mathrm{cr}(\pi^{\mathcal{S}}_{\eta',\bar{\Omega}})$$

To see the existence of such a set Z, we argue as follows. For each limit ordinal $\eta \in [\gamma, \bar{\Omega}]_S$ let

$$\xi_\eta = \text{the least } \xi \text{ such that } \mathrm{cr}(\pi^{\mathcal{S}}_{\eta,\bar{\Omega}}) \in \mathrm{rng}(\pi^{\mathcal{S}}_{\xi,\eta})$$

Note that the function $\eta \mapsto \xi_\eta$ is regressive, so there is a stationary set $S \subseteq [\gamma, \bar{\Omega}]_S$ and a fixed ξ such that for all $\eta \in S$, $\xi_\eta = \xi$. By the definition of ξ_η, for each $\eta \in S$ there is a $\gamma_\eta \in M^{\mathcal{S}}_\xi$ such that $\pi^{\mathcal{S}}_{\xi,\eta}(\gamma_\eta) = \mathrm{cr}(\pi^{\mathcal{S}}_{\eta,\bar{\Omega}})$. Note by the definition of $\alpha_{\mathcal{S}}$ we have

$$\pi^{\mathcal{S}}_{\gamma,\eta}(\alpha_{\mathcal{S}}) = \pi^{\mathcal{S}}_{\xi,\eta} \circ \pi^{\mathcal{S}}_{\gamma,\xi}(\alpha_{\mathcal{S}}) \geq \pi^{\mathcal{S}}_{\xi,\eta}(\gamma_\eta) \implies \pi^{\mathcal{S}}_{\gamma,\xi}(\alpha_{\mathcal{S}}) \geq \gamma_\eta$$

Hence, by the pigeonhole principle, there is a cofinal $Z \subseteq S$ and one fixed γ such that for all $\eta \in Z$, $\gamma_\eta = \gamma$. To summarize, we have an unbounded set Z such that for all $\eta \in Z$, $\pi^{\mathcal{S}}_{\xi,\eta}(\gamma) = \mathrm{cr}(\pi^{\mathcal{S}}_{\eta,\bar{\Omega}})$. We now show that Z is actually closed in $\bar{\Omega}$ which completes the proof that Z is club.

Assume δ is a limit point of Z. We need to see that $\delta \in Z$, that is that $\pi^{\mathcal{S}}_{\xi,\delta}(\gamma) = \mathrm{cr}(\pi^{\mathcal{S}}_{\delta,\bar{\Omega}})$. First, note that $\mathrm{cr}(\pi^{\mathcal{S}}_{\delta,\bar{\Omega}}) > \mathrm{cr}(\pi^{\mathcal{S}}_{\eta,\delta})$ for any $\eta < \delta$ and $\eta \in Z$ as these ordinals are along the same branch. So, for any $\eta < \delta, \eta \in Z$

$$\mathrm{cr}(\pi^{\mathcal{S}}_{\delta,\bar{\Omega}}) > \pi^{\mathcal{S}}_{\xi,\eta}(\gamma) = \mathrm{cr}(\pi^{\mathcal{S}}_{\eta,\bar{\Omega}}) = \mathrm{cr}(\pi^{\mathcal{S}}_{\eta,\delta})$$

Hence, as δ is a limit ordinal along the branch,

$$\mathrm{cr}(\pi^{\mathcal{S}}_{\delta,\bar{\Omega}}) \geq \sup_{\eta \in Z \cap \delta} \pi^{\mathcal{S}}_{\xi,\eta}(\gamma) = \pi^{\mathcal{S}}_{\xi,\delta}(\gamma)$$

On the other hand, if $\mathrm{cr}(\pi^{\mathcal{S}}_{\delta,\bar{\Omega}}) > \pi^{\mathcal{S}}_{\xi,\delta}(\gamma)$ then fix some $\eta \in Z$ with $\delta < \eta$. We then have

$$\mathrm{cr}(\pi^{\mathcal{S}}_{\delta,\bar{\Omega}}) > \pi^{\mathcal{S}}_{\xi,\delta}(\gamma) \implies \mathrm{cr}(\pi^{\mathcal{S}}_{\delta,\bar{\Omega}}) > \pi^{\mathcal{S}}_{\delta,\eta} \circ \pi^{\mathcal{S}}_{\xi,\delta}(\gamma) = \pi^{\mathcal{S}}_{\xi,\eta}(\gamma) = \mathrm{cr}(\pi^{\mathcal{S}}_{\eta,\bar{\Omega}})$$

Which cannot happen as critical points must increase along branches. Hence, $\mathrm{cr}(\pi^{\mathcal{S}}_{\delta,\bar{\Omega}}) = \pi^{\mathcal{S}}_{\xi,\delta}(\gamma)$. So, $\delta \in Z$ and Z is club in $\bar{\Omega}$.

Now, note that if $\eta < \eta' \in Z$ then

$$\mathrm{cr}(\pi^{\mathcal{S}}_{\eta',\bar{\Omega}}) = \pi^{\mathcal{S}}_{\xi,\eta'}(\gamma) = \pi^{\mathcal{S}}_{\eta,\eta'} \circ \pi^{\mathcal{S}}_{\xi,\eta}(\gamma) = \pi^{\mathcal{S}}_{\eta,\eta'}(\mathrm{cr}(\pi^{\mathcal{S}}_{\eta,\bar{\Omega}}))$$

For $\eta \in Z$, let $\iota_\eta = \mathrm{cr}(\pi^{\mathcal{S}}_{\eta,\bar{\Omega}}), \upsilon_\eta = \iota_\eta^{+M^{\mathcal{S}}_\eta} = \iota_\eta^{+P}$ and $B = \{\iota_\eta \mid \eta \in Z\}$. Then, B is club in $\bar{\Omega}$ as the map $\eta \mapsto \iota_\eta$ is normal. Moreover, for $\eta < \eta'$ in Z

$$\sup \pi^{\mathcal{S}}_{\eta,\eta'}[\upsilon_\eta] = \upsilon_{\eta'} = \pi^{\mathcal{S}}_{\eta,\eta'}(\upsilon_\eta) \tag{2.14}$$

by the properties of fine ultrapowers. (If we are taking an n-ultrapower and υ_η is the n^{th} projectum, then it follows by weak-amenability of the extenders.) Recall that as we mentioned above Q is a thick weasel, so there is a club $Z' \subseteq \bar{\Omega}$ such that for all $\iota \in Z' \cap A_0$,

$\iota^{+Q} = \iota^+$. Let $\bar\eta = \min(Z)$ and fix $\iota_\eta \in B - (\iota_{\bar\eta} + 1) \cap Z' \cap A_0$. Then,

$$\iota_\eta^+ = \iota_\eta^{+Q} \leq \iota_\eta^{+P} \leq \iota_\eta^+ \implies \iota_\eta^+ = \iota_\eta^{+Q} = \iota_\eta^{+P} = \upsilon_\eta \tag{2.15}$$

From Equation 2.14 it follows that

$$\mathrm{cf}(\iota_\eta^+) = \mathrm{cf}(\upsilon_\eta) = \mathrm{cf}(\upsilon_{\bar\eta}) \leq \upsilon_{\bar\eta} < \iota_\eta^+ \tag{2.16}$$

So, ι_η^+ is not regular, a contradiction. Thus, we cannot have $Q \lhd P$, and so the coiteration results in the same last model $\bar W^*$.

Now, we would like to see that $\bar W'$ is the root model on the main branch of $\bar{\mathcal{S}}$. For contradiction, assume this is not the case, so $\bar W$ is the root of the main branch of $\bar{\mathcal{S}}$. Let $\Delta = \{\alpha < \bar\Omega \mid \pi^{\mathcal{S}}_{0,\bar\Theta}(\alpha) = \alpha = \pi^{\bar{\mathcal{S}}}_{-1,\bar\Theta}\}$. Then, Δ is a thick class in $\bar W$ and $\bar W^*$. Let E, F be the first extenders used along the main branches of $\mathcal{S}$ and $\bar{\mathcal{S}}$ respectively. We will show that E and F are compatible, a contradiction.

Recall that $\bar W$ is a soundness witness for $\mathsf{K}|\bar\Lambda$ inside H, so it has the hull and definability properties at all $\alpha < \bar\Lambda$. It follows that $\mathrm{cr}(E) = \mathrm{cr}(F)$ as these are the least ordinals $\alpha < \bar\Lambda$ such that $\bar W^*$ does not have the definability property at α. Let $\xi = \mathrm{cr}(E) = \mathrm{cr}(F)$ and $\nu = \min\{\mathrm{lh}(E), \mathrm{lh}(F)\}$. By weak amenability, we have $\mathcal{P}([\xi]^{<\omega}) \cap \bar W = \mathcal{P}([\xi]^{<\omega}) \cap \bar W^*$. Moreover, for $x \in \mathcal{P}([\xi]^{<\omega}) \cap \bar W$, $x = h_{\bar W}(\vec c)$ for some $\vec c \in \Delta$ (where $h_{\bar W}$ is the Σ_1-Skolem function) because $\bar W$ has the hull and definability property at all $\alpha < \bar\Lambda$. Fix $a \in [\nu]^{<\omega}$, $x \in \mathcal{P}([\xi]^{<\omega}) \cap \bar W$ and let $\vec c \in \Delta$ such that $x = h_{\bar W}(\vec c)$. Then,

$$\begin{aligned} x \in E_a \longleftrightarrow a \in \pi^{\mathcal{S}}_{0,\bar\Theta}(x) \longleftrightarrow a \in \pi^{\mathcal{S}}_{0,\bar\Theta}(h_{\bar W}(\vec c)) \longleftrightarrow a \in h_{\bar W^*}(\vec c) \\ \longleftrightarrow a \in \pi^{\bar{\mathcal{S}}}_{-1,\bar\Theta}(h_{\bar W}(\vec c)) \longleftrightarrow x \in F_a \end{aligned}$$

Hence, E and F are compatible which contradicts our assumption that $\bar W$ is the root model of the main branch of $\bar{\mathcal{S}}$. Thus, we have shown that the coiteration results in a final common model $\bar{W}^*$, there is no truncation on the main branches of either side, and $\bar{W}'$ is the root model of the main branch of $\bar{\mathcal{S}}$. $\square$

Let $(\bar{\mathcal{S}}, \mathcal{S})$ continue to denote the coiteration of $(\bar{W}, \bar{W}', \bar{\kappa})$ against $\bar{W}$. Let E be the first extender used on the main branch of $\mathcal{S}$. We show

Lemma 2.21. *$\bar{F}$ is compatible with E, in other words, the normal measure derived from E is $\bar{F}$*

Proof. Let $\bar{\pi} : \bar{W} \to_{\bar{F}} \bar{W}'$ be the ultrapower map and $\pi_1 = \pi^{\bar{\mathcal{S}}}_{0,\bar{\Theta}} : \bar{W}' \to \bar{W}^*$, $\pi_0 = \pi^{\mathcal{S}}_{0,\bar{\Theta}} : \bar{W} \to \bar{W}^*$ be the iteration maps Note, that for $x \in \mathcal{P}(\bar{W}') \cap \bar{\kappa} = \mathcal{P}(\bar{W}) \cap \bar{\kappa}$,. The following diagram summarizes the situation

$$\begin{tikzcd}
\bar{W}' \arrow[dr, "\pi_1"] & \\
\bar{W} \arrow[u, "\bar{\pi}"] \arrow[r, "\pi_0"] & \bar{W}^*
\end{tikzcd}$$

Note that by the rules in the phalanx iteration, $\mathrm{cr}(\pi_1) \geq \bar{\kappa}$. It follows that the map $\pi_1 \circ \bar{\pi} : \bar{W} \to \bar{W}^*$ has critical point $\bar{\kappa}$ as $\mathrm{cr}(\bar{\pi}) = \bar{\kappa}$. As in the previous lemma, because $\bar{W}$ has the hull and definability property for all $\alpha < \bar{\Lambda}$, it follows that $\mathrm{cr}(\pi_0) = \bar{\kappa}$ as well. Moreover, for $x \in \bar{W}|\bar{\Lambda}$, $x = h_{\bar{W}}(\vec{c})$ for some $\vec{c} \in \Delta$, so $\pi_0(x) = \pi_1 \circ \bar{\pi}(x)$. Additionally, (again because $\mathrm{cr}(\pi_1) \geq \bar{\kappa}$), $\mathcal{P}(\bar{W}') \cap \bar{\kappa} = \mathcal{P}(\bar{\kappa}) \cap \bar{W}^*$. Similarly, because $\mathrm{cr}(\pi) = \bar{\kappa}$, we have $\mathcal{P}(\bar{W}) \cap \bar{\kappa} = \mathcal{P}(\bar{\kappa}) \cap \bar{W}^*$. Hence, $\mathcal{P}(\bar{\kappa}) \cap \bar{W} = \mathcal{P}(\bar{\kappa}) \cap \bar{W}'$ and in particular, $\bar{F}$ is weakly amenable with respect to $\bar{W}$.

Claim 2.21.1. $\mathrm{cr}(\pi_1) > \bar{\kappa}$

Proof of Claim 2.21.1. Suppose not, so $\mathrm{cr}(\pi_1) = \bar{\kappa}$. Note that for $x \in \mathcal{P}(\bar{\kappa}) \cap \bar{W} = \mathcal{P}(\bar{\kappa}) \cap \bar{W}$

we have $x = \bar{\pi}(x) \cap \bar{\kappa}$, so

$$\pi_1(x) = \pi_1(\bar{\pi}(x) \cap \bar{\kappa}) = \pi_1 \circ \bar{\pi}(x) \cap \pi_1(\bar{\kappa}) = \pi_0(x) \cap \pi_1(\bar{\kappa}) \tag{2.17}$$

Let $\bar{E}$ be the first extender used on the main branch of $\bar{\mathcal{S}}$ and $\nu = \min\{\mathrm{lh}(E), \mathrm{lh}(\bar{E})\}$. For $x \in \mathcal{P}(\bar{\kappa}) \cap \bar{W} = \mathcal{P}(\bar{\kappa}) \cap \bar{W}$ and $a \in [\nu]^{<\omega}$ we have

$$x \in \bar{E}_a \longleftrightarrow a \in \pi_1(x) \longleftrightarrow a \in \pi_0(x) \cap \pi_1(\bar{\kappa}) \longleftrightarrow x \in E_a \tag{2.18}$$

So, E and $\bar{E}$ are compatible, contradiction. $\square$

Now that we know $\mathrm{cr}(\bar{\pi}) > \bar{\kappa}$, we can show that $\bar{F}$ is compatible with E. Let $y \in \mathcal{P}(\bar{\kappa}) \cap \bar{W}$. So, there is some $\vec{c} \in \Delta$ such that $y = h_{\bar{W}}(\vec{c})$. Because $\bar{F}$ is a normal measure by Lemmas 2.14 and 2.17 we then have

$$\begin{aligned} y \in \bar{F} &\longleftrightarrow \bar{\kappa} \in \bar{\pi}(y) \longleftrightarrow \bar{\kappa} \in \bar{\pi}(h_{\bar{W}}(\vec{c})) \longleftrightarrow \bar{\kappa} \in \pi_1 \circ \bar{\pi}(h_{\bar{W}}(\vec{c})) \\ &\longleftrightarrow \bar{\kappa} \in h_{\bar{W}^*}(\vec{c}) \longleftrightarrow \bar{\kappa} \in \pi_0(h_{\bar{W}}(\vec{c})) \longleftrightarrow \bar{\kappa} \in \pi_0(y) \longleftrightarrow y \in E_{\{\bar{\kappa}\}} \end{aligned}$$

The equivalences on the first line holds because $\mathrm{cr}(\pi_1) > \bar{\kappa}$ and the equivalences on the second line hold because $\vec{c} \in \Delta$. Hence, $\bar{F}$ is compatible with E which proves the lemma.

$\square$

Finally we are in a position to prove Theorem 2.3

Proof of Theorem 2.3. By Lemma 2.21 there is an extender E on the $\bar{W}$-sequence which is compatible with $\bar{F}$. In particular, E has critical point $\bar{\kappa}$. Via the map $\sigma : \bar{W} \to W$, we can map E to $E' = \sigma(E)$, which will be an extender on the W-sequence with critical point κ. Because W is a soundness witness for for $\mathsf{K} \mid \Lambda$ and $\kappa < \Lambda$, E' is also on the K-sequence.

Thus, κ is measurable in K. $\square$

Next, we prove Theorem 2.22.

Theorem 2.22. *Assume $\kappa > \omega_2$ is a regular cardinal in K, but $\mathrm{cf}(\kappa) < \mathrm{card}(\kappa)$. Assume $\gamma = \mathrm{cf}(\kappa) > \omega$ and $\mathrm{card}(\kappa)$ is γ-closed. Then, $o^{\mathsf{K}}(\kappa) \geq \gamma$, where $o^{\mathsf{K}}(\kappa)$ is the Mitchell order of κ in K.*

The proof of Theorem 2.22 is similar to the proof of Theorem 2.3 and many of the previous lemmas are still true. As in Theorem 2.3, we let W be a soundness witness for $\mathsf{K}|\Lambda$ for a $\Lambda >> \kappa$. Freeing the notation from the proof of Theorem 2.3, construct an elementary substructure X with all the relevant information. Fix $\kappa > \omega_2$ as in the statement of the theorem. Build an elementary substructure $X \prec (H_\Gamma, \in)$ where $\Gamma >> \Omega$ such that

- $\mathrm{card}(X) < \mathrm{card}(\kappa)$
- $\sup(X \cap \kappa) = \kappa$
- ${}^{\gamma}X \subseteq X$
- $\Omega, \mu, W, etc \in X$

We are able to build such an X based on our assumption that $\mathrm{card}(\kappa)$ is γ-closed. Let $\sigma_0 : H \to X$ be the inverse to the transitive collapse. Define $\bar{W} = \sigma_0^{-1}(W)$ and $\sigma : \bar{W} \to W = \sigma_0 \restriction \bar{W}$. Let $(\bar{\mathcal{T}}, \mathcal{T})$ be the coiteration of $(\bar{W}|\bar{\kappa}, W)$

Fact 2.5, Lemma 2.7, and Lemma 2.8 hold here, and the next lemma summarizes everything relevant in this next context.

Lemma 2.23. *1. Let $\Theta + 1$ be the length of the coiteration $(\bar{\mathcal{T}}, \mathcal{T})$. Then, $\bar{W}|\bar{\kappa}$ does not move, $\bar{W}|\bar{\kappa} \lhd W_\Theta$, and there is a truncation along the main branch of $\mathcal{T}$.*

2. *$\bar{\kappa}$ is not definably singularized over W_Θ*

3. *The sequence of critical points $(\kappa_i \mid i+1 \in (0,\Theta)_T)$ of extenders used along the main branch of $\mathcal{T}$ is cofinal in $\bar{\kappa}$*

Because $\mathrm{cf}(\Theta) = \gamma > \omega$ in this case, Lemma 2.9 can be improved to

Lemma 2.24. *The set $C_0 \subseteq (0,\Theta)_T$, defined by*

$$C_0 = \{\xi \in (0,\Theta)_T \mid \pi_{\xi,\Theta}(\mathrm{cr}(\pi_{\xi,\Theta})) = \bar{\kappa}\}$$

is club in Θ.

Proof. By Lemma 2.9, we already know C_0 is unbounded in Θ. We simply need to show that C_0 is closed as well. So, let $\delta < \Theta$ be a limit point of C_0. By construction of the iteration tree, $\delta \in (0,\Theta)_T$ i.e. it is along the main branch. Recall, that for $\xi_i \in C_0$, we have $\mathrm{cr}(\pi_{\xi_i,\Theta}) = \kappa_i$ and $\pi_{\xi_i,\xi_j}(\kappa_i) = \kappa_j$ for any $\xi_j \in C_0$ with $\xi_j > \xi_i$. We claim that

$$\mathrm{cr}(\pi_{\delta,\Theta}) = \pi_{\xi_i,\delta}(\kappa_i) \quad \text{for } \xi_i \in C_0 \cap \delta$$

Note that for $\xi_i \in C_0$, $\pi_{\xi_i,\delta}(\kappa_i) = \sup\{\kappa_i \mid \xi_i \in C_0 \cap \delta\}$. This is because δ is a limit point on the branch and the critical points map to each other along the branch. Clearly, $\mathrm{cr}(\pi_{\delta,\Theta})$ is greater than this supremum as critical points strictly increase along iteration branches. So, $\mathrm{cr}(\pi_{\delta,\Theta}) \geq \pi_{\xi_i,\delta}(\kappa_i)$ for any $\xi_i \in C_0 \cap \delta$. Suppose for contradiction that we did not have equality. Then, $\mathrm{cr}(\pi_{\delta,\Theta}) > \pi_{\xi_i,\delta}(\kappa_i)$ for $\xi_i \in C_0 \cap \delta$. However, if $\xi_j \in C_0$ is such that $\xi_j > \delta$, then $\pi_{\xi_i,\xi_j}(\kappa_i) = \kappa_j$, so that

$$\kappa_j = \pi_{\xi_i,\xi_j}(\kappa_i) = \pi_{\delta,\xi_i} \circ \pi_{\xi_i,\xi_j}(\kappa_i) < \mathrm{cr}(\pi_{\delta,\Theta})$$

an obvious contradiction.

So, $\mathrm{cr}(\pi_{\delta,\Theta}) = \pi_{\xi_i,\delta}(\kappa_i)$ for $\xi_i \in C_0 \cap \delta$. But then, it is obvious to see that $\delta \in C_0$ as we have

$$\pi_{\delta,\Theta}(\mathrm{cr}(\pi_{\delta,\Theta})) = \pi_{\delta,\Theta}(\pi_{\xi_i,\delta}(\kappa_i)) = \pi_{\xi_i,\Theta}(\kappa_i) = \bar{\kappa} \quad \text{for } \xi_i \in C_0 \cap \delta$$

Hence, C_0 is closed. □

Remark 2.25. We remark that it is also not hard to see that actually for δ a limit point of C_0, we have $\kappa_\delta = \mathrm{cr}(\pi_{\delta,\Theta})$ and that E_δ is applied to W_δ.

As H is closed under γ-sequences, let us fix a set $C_1 \subseteq C_0$ which is club in Θ and such that $C_1 \in H$. Without loss of generality, assume $C_1 \subseteq (\varepsilon, \Theta)_T$ where $\varepsilon + 1 \in (0, \Theta)_T$ is past any truncations or drops in degree. We next prove

Lemma 2.26. *For each $\beta < \gamma$, the set*

$$C_\beta = \{\xi_i \in C \mid o_*^{W_i}(E_i) \geq \beta\}$$

contains a club in Θ.

Proof. Suppose this is not the case. Fix $\beta < \gamma$ such that C_β does not contain a club. It follows that $S_0 = \{\xi_i \in C \mid o_*^{W_i}(E_i) < \beta\}$ is stationary. By Fodor's lemma, there is a stationary set $S \subseteq S_0$ and a fixed $\gamma' < \beta$ such that for all $\xi_i \in S$, $o_*^{W_i}(E_i) = \gamma'$. By shrinking S if necessary, assume without loss of generality that $\kappa_i > \gamma'$ for all $\xi_i \in S$. Then, for $\xi_i \in S$ E_i is the trivial completion of a measure as it has order γ'. So, E_i is applied to W_i, in other words $\xi_i = i$. Moreover, for $i, j \in S$ with $i < j$, $\pi_{i,j}(E_i) = E_j$.

Fix an ordinal $\zeta \in S$ which is also a limit point of S, so E_ζ has order γ' and $\pi_{i,\zeta}(E_i) = E_\zeta$ for $i \in S$ with $i < \zeta$. Moreover, as in Remark 2.25, E_ζ is applied to W_ζ. We claim that for $x \in \mathcal{P}(\kappa_\zeta) \cap W_\zeta$ we have

$$x \in (E_\zeta)_{\{\kappa_\zeta\}} \longleftrightarrow (\exists j \in S \cap \zeta)\ \kappa_i \in x \text{ for all } i \in S \cap \zeta - j \tag{2.19}$$

To see this equivalence, fix $x \in \mathcal{P}(\kappa_\zeta) \cap W_\zeta$ and let $j \in S \cap \zeta$ large enough and $\bar{x} \in W_j$ so that $\pi_{j,\zeta}(\bar{x}) = x$. Then, because critical points increase along branches we have

$$x \in (E_\zeta)_{\{\kappa_\zeta\}} \longleftrightarrow \bar{x} \in (E_j)_{\{\kappa_j\}} \longleftrightarrow \kappa_j \in \pi_{j,\zeta}(\bar{x}) = x$$

Because E_j and κ_j are mapped to E_i and κ_i via the iteration map $\pi_{j,i}$ for $i \in S$ and $i > j$ we have

$$x \in (E_\zeta)_{\{\kappa_\zeta\}} \longleftrightarrow \bar{x} \in (E_j)_{\{\kappa_j\}} \longleftrightarrow \pi_{j,i}(\bar{x}) \in E_i \longleftrightarrow \kappa_i \in \pi_{j,\zeta}(\bar{x}) = x$$

which proves Equation 2.19. We next prove

Claim 2.26.1. E_ζ is on the $\bar{W}$-sequence

Proof of Claim 2.26.1. Since $\bar{W}|\bar{\kappa} \lhd W_\Theta$ and $\bar{\kappa}$ is a cardinal in W_Θ, $\mathcal{P}(\kappa_\zeta) \cap W_\zeta = \mathcal{P}(\kappa_\zeta) \cap W_\Theta = \mathcal{P}(\kappa_\zeta) \cap \bar{W}$ by acceptability. So, $(E_\zeta)_{\{\kappa_\zeta\}}$ is a weakly amenable normal measure over $\bar{W}$. Because $S, (\kappa_i \mid i \in S) \in H$ and H is closed under γ-sequences we can reconstruct the normal measure $E = (E_\zeta)_{\{\bar{\kappa}\}}$ inside H. We will now work inside H and proceed in similar fashion as Lemmas 2.20 and 2.21.

Inside H, let $\bar{W}' = \mathrm{Ult}(\bar{W}, E)$ as in Lemma 2.19, this ultrapower is well founded and hence iterable by Corollary 15 of [1]. So, the phalanx $(\bar{W}, \bar{W}', \kappa_\zeta)$ is iterable because it can be embedded into $\bar{W}'$. We compare $(\bar{W}, \bar{W}', \kappa_\zeta)$ against $\bar{W}$.

Letting $(\bar{\mathcal{U}}, \mathcal{U})$ be the coiteration of $(\bar{W}, \bar{W}', \kappa_\zeta)$ with $\bar{W}$, the same argument as in Lemma 2.20 proves that $\bar{W}'$ is the root model on the main branch of $\bar{\mathcal{U}}$, there are no truncations on the main branch of either tree, and they coiterate to a common last model $\bar{W}^*$.

Let F be the first extender used along the branch $\bar{W}$-to-$\bar{W}^*$. Using the hull and definability properties, as in Lemma 2.21 we conclude that F is compatible to E. Notice, then that F

has critical point κ_ζ and the normal measure derived from F and κ_ζ is just E. As F is on the $\bar{W}$-sequence, it follows that the trivial completion of E is on the $\bar{W}$-sequence by Lemma 1.76. However, the trivial completion of E is E_ζ, so that E_ζ is on the $\bar{W}$-sequence. $\square$

So E_ζ is on the $\bar{W}$-sequence. But E_ζ is used on the coiteration of W against $\bar{W}|\bar{\kappa}$, a contradiction. This completes the proof that for every $\beta < \gamma$ the set $C_\beta \subseteq \Theta$ contains a club. $\square$

For each $\beta < \gamma$ let $C_\beta \subseteq C_1$ be a set guaranteed by the previous lemma such that $C_\beta \in H$. For $\beta < \gamma$ and each $\xi_i \in C_\beta$, let F_i^β be the extender on the W_i-sequence such that $o_*^{W_i}(F_i^\beta) = \beta$. Fix $\beta < \gamma$. Because $\gamma < \bar{\kappa}$ and $(\kappa_i \mid \xi_i \in C_\beta)$ is cofinal in $\bar{\kappa}$, without loss of generality, we can assume that $\kappa_i > \beta$ for all $\xi_i \in C_\beta$ (shrinking C_β if necessary). As before, we have

Lemma 2.27. *For each $\beta < \gamma$ and $\xi_i \in C_\beta$, F_i^β is on the $\widehat{W_{\xi_i}}$-sequence*

Proof. Note that each F_i^β is just the trivial completion of a measure, so if $\alpha = \mathrm{lh}(F_i^\beta)$ then $\alpha < \tau_i^{+W_i}$. So the same argument as in Lemma 2.15 works here. $\square$

Hence, for $\xi_i, \xi_j \in C_\beta$ with $\xi_i < \xi_j$ we have $\pi_{\xi_i,\xi_j}(F_i^\beta) = F_j^\beta$. For $\beta < \gamma$, we let $F^\beta = \pi_{\xi_i,\Theta}(F_i^\beta)$ where $\xi_i \in C_\beta$. So, F^β is an extender on the W_Θ-sequence and $o_*^{W_\Theta}(F^\beta) = \beta$.

Lemma 2.28. *For $\beta < \gamma$, and $x \in \mathcal{P}(\bar{\kappa}) \cap W_\Theta$ we have*

$$x \in (F^\beta)_{\{\bar{\kappa}\}} \longleftrightarrow (\exists j \in C_\beta)\ x \cap \kappa_i \in F_i^\beta \textit{ for all } \xi_i \in C_\beta - j$$

Proof. This is of course similar to Lemma 2.16. Given $x \in \mathcal{P}(\bar{\kappa}) \cap W_\Theta$ let $\xi_i \in C_\beta$ be large enough such that there is $\bar{x} \in W_{\xi_i}$ with $x = \pi_{\xi_i,\Theta}(\bar{x})$. Then, noting that $\bar{x} = \pi_{\xi_i,\Theta}(\bar{x}) \cap \kappa_i = x \cap \kappa_i$ we have

$$x \in (F^\beta)_{\{\bar{\kappa}\}} \longleftrightarrow \bar{x} \in (F_i^\beta)_{\{\kappa_i\}} \longleftrightarrow x \cap \kappa_i \in (F_i^\beta)_{\{\kappa_i\}} \tag{2.20}$$

Similar to the previous arguments, Equation 2.20 holds for all larger $\xi_j \in C_\beta$ as κ_i and F_i^β are mapped to κ_j and F_j^β respectively via π_{ξ_i,ξ_j}. □

Inside H we now define measures $\bar{F}^\beta$ on $\mathcal{P}(\bar{\kappa}) \cap \bar{W}$ following this equivalence. This is possible because each $C_\beta \in H$ and each F_i^β for $\xi_i \in C_\beta$ is on the $\bar{W}$-sequence by coherence. For $x \in \mathcal{P}(\bar{\kappa}) \cap \bar{W}$, set

$$x \in \bar{F}^\beta \longleftrightarrow (\exists j \in C_\beta)\ x \cap \kappa_i \in (F_i^\beta)_{\{\kappa_i\}} \text{ for all } \xi_i \in C_\beta - j \tag{2.21}$$

As in Lemma 2.14 we get

Lemma 2.29. *For each $\beta < \gamma$, $\bar{F}^\beta$ is normal $\bar{\kappa}$-complete ultrafilter on $\mathcal{P}(\bar{\kappa}) \cap \bar{W}$.*

Proof. See Lemma 2.14. □

Inside H, let $\bar{W}'_\beta = \mathrm{Ult}(\bar{W}, \bar{F}^\beta)$ for $\beta < \gamma$. This ultrapower is well founded because each $\bar{F}^\beta$ is ω-complete. Hence, as in Lemma 2.18, $\bar{W}'_\beta$ is fully iterable. It follows that the phalanx $(\bar{W}, \bar{W}'_\beta, \bar{\kappa})$ is iterable as well. Let $(\bar{\mathcal{S}}, \mathcal{S})$ be the coiteration of $(\bar{W}, \bar{W}'_\beta, \bar{\kappa})$ against $\bar{W}$ We show

Lemma 2.30. *For $\beta < \gamma$, $\bar{F}^\beta$ is compatible to an extender E_β on the $\bar{W}$-sequence*

Proof. Following Lemma 2.20, we can show that there are no truncations on the main branch of either tree, $\bar{W}'_\beta$ is the root model on the phalanx side, and the coiteration results in a common final model. Then, letting E_β be the first extender used on the main branch of $\mathcal{S}$, using an argument similar to that of Lemma 2.21, we can show that $\bar{F}^\beta$ is compatible to E_β.

□

Finally, we show

Lemma 2.31. *For $\beta < \gamma$, we have $o^{\bar{W}}(E^\beta) \geq \beta$, where $o^{\bar{W}}$ is the Mitchell order.*

Proof. For each $\delta < \gamma$, define a function $f_\delta : \bar{\kappa} \to \bar{\kappa} \in \bar{W}$ by

$$f_\delta(\xi) = \begin{cases} \mathrm{lh}(E_\zeta^{\bar{W}}) \text{ if } E_\zeta^{\bar{W}} \text{ is the unique total extender s.t. } \mathrm{cr}(E_\zeta^{\bar{W}}) = \xi, o_*^{\bar{W}}(E_\zeta^{\bar{W}}) = \delta \\ 0 \text{ if no such } E_\zeta^{\bar{W}} \text{ exists} \end{cases}$$

Note that because $\bar{W}|\bar{\kappa} \lhd W_\Theta$, $f_\delta \in W_\Theta$ and f_δ has the same definition over W_Θ with W_Θ replacing $\bar{W}$ everywhere. Fix $\beta < \gamma$ and recall that F^β is on the W_Θ-sequence, $\mathrm{cr}(F^\beta) = \bar{\kappa}$, and $o_*^{W_\Theta}(F^\beta) = \beta$. It follows that by coherence, if $\delta < \beta$, then F^δ is on the $W'_\beta = \mathrm{Ult}(W_\Theta, F^\beta)$-sequence. Moreover, $o_*^{W'_\beta}(F^\delta) = \delta$. Hence, letting $\pi : W_\Theta \to W'_\beta$ be the ultrapower map, $\mathrm{lh}(F^\delta) = \pi(f_\delta)(\bar{\kappa})$. Hence, if $\delta_1 < \delta_2 < \beta$, then $\mathrm{lh}(F^{\delta_1}) < \mathrm{lh}(F^{\delta_2})$, so

$$z_{\delta_1,\delta_2} = \{\xi < \bar{\kappa} \mid f_{\delta_1}(\xi) < f_{\delta_2}(\xi)\} \in F^\beta_{\{\bar{\kappa}\}} \tag{2.22}$$

Because F^β is compatible to E^β it follows that $z_{\delta_1,\delta_2} \in E^\beta_{\{\bar{\kappa}\}}$. Hence, letting $\bar{\pi} : \bar{W} \to \mathrm{Ult}(\bar{W}, E^\beta)$ be the ultrapower map, for $\delta_1 < \delta_2 < \beta$ we have $\bar{\pi}(f_{\delta_1})(\bar{\kappa}) < \bar{\pi}(f_{\delta_2})(\bar{\kappa})$. So, for each $\delta < \beta$, $\bar{\pi}(f_\delta)(\bar{\kappa})$ is the length of a total extender on $\bar{\kappa}$ on the $\mathrm{Ult}(\bar{W}, E^\beta)$-sequence with order equal to δ. It follows that $o^{\bar{W}}(E^\beta) > \delta$ for each $\delta < \beta$, so $o^{\bar{W}}(E^\beta) \geq \beta$. □

Finally, we complete the proof of Theorem 2.22.

Proof of Theorem 2.22. We have shown that for each $\beta < \gamma$, there is a total extender E^β on the $\bar{W}$-sequence with critical point $\bar{\kappa}$ and such that $o^{\bar{W}}(E^\beta) \geq \beta$. Recall $\sigma : \bar{W} \to W$ is elementary, so $\sigma(E^\beta)$ is a total extender on the W-sequence with critical point κ and $o^W(\sigma(E^\beta)) \geq \beta$. It follows that $o^W(\kappa) \geq \gamma$. Because W and K agree up to $\Lambda >> \kappa$ it follows that $o^{\mathsf{K}}(\kappa) \geq \gamma$ as well. □

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